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Scaling functions from q-deformed Virasoro characters

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Abstract
We propose a renormalization group (RG) scaling function which is constructed from q-deformed fermionic versions of Virasoro characters. By comparison with alternative methods, which take their starting point in the massive theories, we demonstrate that these new functions contain qualitatively the same information. We show that these functions allow for RG-flows not only amongst members of a particular series of conformal field theories, but also between different series such as $N = 0, 1, 2$ supersymmetric conformal field theories. We provide a detailed analysis of how Weyl characters may be utilized in order to solve various recurrence relations emerging at the fixed points of these flows. The q-deformed Virasoro characters allow furthermore for the construction of particle spectra, which involve unstable pseudo-particles.

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1. Introduction

Renormalization group (RG) methods have been developed [1] to carry out qualitative studies of regions of quantum field theories which are not accessible to perturbation theory in the coupling constant. For theories in 1 + 1 space–time dimensions these methods admit particularly powerful realizations in the form of explicit constructions of scaling functions. Such functions may be obtained from the thermodynamic Bethe ansatz (TBA) [2], from correlation functions involving various components of the energy–momentum tensor [3, 4] or from semiclassical studies [5]. In general, the functions obtained from different approaches differ quantitatively, but nonetheless possess the same qualitative features. It is clear that they are different, since their conceptual origin is also not the same. In the TBA-context, these functions have been coined to be scaling functions reflecting the very fact that the characterizing parameter controls the size of the system. This is, of course, different from the

c-theorem formulation in [3, 4], where the RG-parameter is related to the distance between the two operators in the correlation function. Despite this all these functions share some common features which can be characterized as follows.

We consider a unitary quantum field theory which contains asymptotically stable particles of mass m_i and unstable particles of mass M_i. In addition, we assume that there are no particles associated with asymptotic massless states in the spectrum. Then the scaling function $c(r)$ parametrized by a dimensionless RG-parameter r has the following properties:

(i) It coincides with the Virasoro central charge c of the ultraviolet conformal field theory for vanishing r,

$$\lim_{r\to 0} c(r) = c. \tag{1}$$

(ii) It is non-increasing along the RG-flow.

(iii) It is stationary at RG fixed points and acquires at these points the Virasoro central charge of specific conformal field theories,

$$c(r) = c_{ij} = \text{const} \qquad m_i, M_i \ll \frac{2}{r} \ll m_j, M_j. \tag{2}$$

(iv) It vanishes in the infrared,

$$c(r) = 0 \qquad \frac{2}{r} \ll m_i, M_i. \tag{3}$$

There is yet another proposal to construct such type of functions, namely as 'Bailey flow' [6] between different series of Virasoro characters. However, so far it has neither been established whether the functions constructed in this fashion satisfy properties (i)–(iv) nor has it been clarified in which way they are related to a massive quantum field theory.

In the following we shall be constructing a scaling function which also flows between certain Virasoro characters. In addition to the flows provided in [6], we will not only propose a flow between several distinct series, such as for instance from $N = 2$ superconformal theories to $N = 1$ superconformal theories, but also realize the flows within a particular series itself. Our flows are manifested by means of q-deformed Cartan matrices which simulate a control of the energy scales of unstable particles. We establish that the proposed function indeed satisfies properties (i)–(iv) and in addition relates it to a concrete massive quantum field theory with an explicitly known scattering matrix.

Our manuscript is organized as follows: in section 2 we recall how certain recurrence relations emerge from a saddle point analysis of fermionic versions of Virasoro characters, which involve data of the massive theory, namely the phase of the scattering matrix, and how their solutions are related to the effective central charge. We show that various series may be realized in terms of the homogeneous sine–Gordon (HSG) models. In section 3 we present a q-deformed version of the analysis in section 2 and demonstrate how the HSG realization allows for a flow amongst various models governed by the mass scales of the unstable particles. The analysis in this section is mainly carried out numerically. Section 4 is devoted to the explicit analytic solutions at the plateaux in terms of principally specialized Weyl characters. We present here various cases which have not been considered before. In section 5 we demonstrate how the q-deformed characters may be associated with particle spectra, which also involve unstable pseudo-particles. Our conclusions are stated in section 6.

2. The TBA from the massive and massless sides

Let us first recall some well-known facts in order to assemble the relevant equations and to establish our notation. We consider a Virasoro character in the so-called 'fermionic version'[1] [7]

$$\chi(q)=\sum_{\vec{m}\in\mathfrak{S}}^{\infty}q^{\vec{m}M\vec{m}^t/2+\vec{m}\cdot\vec{B}}\prod_{i=1}^{l}\begin{bmatrix}(\vec{m}(1-M))_i+B'_i\\ m_i\end{bmatrix}_q. \tag{4}$$

Here we employ the standard abbreviation for Euler's function $(q)_m$ with $(q)_0=1$ and the Gaußian polynomial (q-binomial), see e.g. [8], for the integers n and m with $0\leqslant m\leqslant n$,

$$(q)_m:=\prod_{k=1}^{m}(1-q^k)\qquad\text{and}\qquad\begin{bmatrix}n\\ m\end{bmatrix}_q:=\frac{(q)_n}{(q)_m(q)_{n-m}}. \tag{5}$$

The main characteristics of expression (4) for the character $\chi(q)$ are the real symmetric $(l\times l)$-matrix M and the vector $\vec{B}'$ with $B'_i=\infty$ for $1\leqslant i\leqslant l-l'$, $B'_i=0$ for $l-l'<i\leqslant l$, with l' being a non-negative integer smaller than l. The specific form of the vector $\vec{B}$ distinguishes between different highest weight representations, which share of course the same Virasoro central charge c. There might be restrictions on the set $\mathfrak{S}$ in which $\vec{m}$ takes its values, which usually reflect some of the symmetries in the model.

The important thing for us to note is that once $\chi(q)$ is of the generic form (4), one may employ the techniques originally pursued in [9] and carry out a saddle point analysis to extract the leading order behaviour. As a result of this, the effective central charge, i.e. $c_{\text{eff}}=c-24h'$ with h' being the smallest conformal dimension occurring in the theory ($h'=0$ in unitary models), is expressed in a rather non-obvious way. For the character of the particular form (4), this analysis was performed first in [7], leading, after a suitable variable transformation, to the saddle point conditions

$$1-x_A=\prod_{B=1}^{l}(x_B)^{M_{AB}}\qquad\text{and}\qquad 1-y_A=\prod_{B=1+l-l'}^{l}(y_B)^{M'_{AB}}. \tag{6}$$

At this stage x_A and y_A are just the integration variables occurring in this context (for details see e.g. [7, 9]). The matrix M' is a submatrix of M of dimension $(l'\times l')$. The remaining y's which do not occur in these equations are taken to be 1, i.e. $y_A=1$ for $1\leqslant A\leqslant l-l'$. One should also note that, since in this analysis sums are converted into integrals, the specific structure of the set $\mathfrak{S}$ does not affect the outcome of the computation and may therefore be ignored for our purposes. The leading order behaviour at the extremum point yields the effective central charge

$$c_{\text{eff}}=\frac{6}{\pi^2}\sum_{A=1}^{l}(\mathcal{L}(1-x_A)-\mathcal{L}(1-y_A)) \tag{7}$$

in terms of Rogers dilogarithm $\mathcal{L}(x)=\sum_{n=1}^{\infty}x^n/n^2+\ln x\ln(1-x)/2$ (for properties see e.g. [10]). Once c_{eff} is rational, system (6) and (7) is referred to as 'accessible' dilogarithms (for a review see e.g. [11] and references therein), which from the mathematical point of view is a rather exceptional situation.

The important point to note here is that the saddle point analysis does not rely upon the fact that the matrices M and M' are constant. It is this feature which we shall exploit below.

[1] In fact this terminology is slightly misleading, since they are not intrinsically fermionic. This name originated from the construction of fermionic pseudo-particle spectra. However, it is also possible to construct from (4) pseudo-particle spectra related to all kinds of general statistics.

2.1. $\mathbf{g}|\tilde{\mathbf{g}}$-*theories*

Intriguingly the same set of equations (6) and (7) may also be obtained when we commence with the massive instead of the conformal side. We start from a scattering matrix $S_{AB}(\theta)$, as a function of the rapidity, θ, between particles of type $1 \leqslant A, B \leqslant l$. Performing then a thermodynamic Bethe ansatz analysis [2] one ends up with a set of non-linear integral equations in the pseudo-energies as functions of the rapidities, the so-called TBA-equations. We then assume that the S-matrix is such that it leads to regions in the TBA-equations in which the pseudo-energies are constant. In general, this happens when the scattering matrix does not depend on the effective coupling constant. In that situation, the thermodynamic Bethe ansatz leads to a set of coupled equations coinciding precisely with the ones in x in (6). All y's may be thought of as being 1 in this case. The matrix M in (4) is now directly related to the massive models containing the information about the scattering matrix,

$$M_{AB} = \delta_{AB} - \frac{1}{2\pi\mathrm{i}} \lim_{\theta\to\infty} \ln(S_{AB}(\theta)S_{BA}(\theta)). \tag{8}$$

Reversing the argument, relation (8) means that one has identified a quantity within the conformal field theory which carries the data of the phase of the S-matrix.

In the following we will consider theories in which M_{AB} is related to a Lie algebraic structure. For this purpose we give the quantum numbers A, B, which describe the particle type, an additional substructure. We identify each particle by two quantum numbers, i.e. $A = (a, i)$, such that the scattering matrices are of the general form $S_{ab}^{ij}(\theta)$. We associate the main quantum numbers a, b to the vertices of the Dynkin diagram of a simply laced Lie algebra $\mathbf{g}$ of rank ℓ and the so-called colour quantum numbers i, j to the vertices of the Dynkin diagram of a simply laced Lie algebra $\tilde{\mathbf{g}}$ of rank $\tilde{\ell}$. We refer to these theories as $\mathbf{g}|\tilde{\mathbf{g}}$. The S-matrices constructed in [12] are of the type

$$S_{ab}^{ij}(\theta) = \mathrm{e}^{\mathrm{i}\pi\varepsilon_{ij}K_{\bar{a}b}^{-1}} \exp\int_{-\infty}^{\infty}\frac{\mathrm{d}t}{t}\left(2\cosh\frac{\pi t}{h} - \tilde{I}\right)_{ij}\left(2\cosh\frac{\pi t}{h} - I\right)_{ab}^{-1} \mathrm{e}^{-\mathrm{i}t(\theta+\sigma_{ij})} \tag{9}$$

with $I, \tilde{I}$ being the incidence matrix of $\mathbf{g}, \tilde{\mathbf{g}}$, respectively. Here ε_{ij} is the Levi-Civita pseudotensor, h is the Coxeter number of $\mathbf{g}$ and $\sigma_{ij} = -\sigma_{ij}$ are the resonance parameters. As special cases of this S-matrix we have the $\mathbf{g}|\mathbf{A}_1$ and $\mathbf{A}_n|\tilde{\mathbf{g}}$ theories which correspond to the minimal affine Toda theories and the $\tilde{\mathbf{g}}_{n+1}$-HSG models [13]. As may be seen easily from (9), the M-matrix for these models is

$$M_{ab}^{ij} = K_{ab}^{-1}\tilde{K}_{ij} \tag{10}$$

with $K, \tilde{K}$ being the Cartan matrices of $\mathbf{g}$, $\tilde{\mathbf{g}}$, respectively (see also [12]). The special case $\mathbf{g}|\mathbf{A}_1$ was first treated in [14]. S-matrices for $\tilde{\mathbf{g}}$ also to be non-simply laced were proposed in [15]. It remains an open question, apart from $\mathbf{g}|\mathbf{A}_1$, how to allow also $\mathbf{g}$ to be non-simply laced.

2.2. $\mathbf{g}|\tilde{\mathbf{g}}$-*coset theories*

The full system (6) and (7), involving a non-trivial M'-matrix, can be associated in general with a non-diagonal scattering matrix on the massive side. A straightforward identification between M and the scattering matrix such as in (8) is not possible in this case. However, within the thermodynamic Bethe ansatz analysis the equations are diagonalized and decoupled, such that at the fixed points they acquire precisely the form (6). In many prominent cases the M and M' matrices involve Lie algebraic quantities in the form of (10). Noting this point, many models can be realized formally in terms of $\mathbf{g}|\tilde{\mathbf{g}}$-cosets.

2.2.1. Unitary minimal models. The series of unitary minimal models, usually denoted by $\mathcal{M}(k, k+1)$ [16], constitute an extremely well-studied and prominent class of conformal field theories. It is well known [17] that they may, for instance, be realized by the cosets $SU(2)_k \otimes SU(2)_1/SU(2)_{k+1}$ or $SU(k+1)_2/SU(k)_2 \otimes U(1)$, which are related to each other by level-rank duality [18]. Recalling the fact [17] that each extended simple Lie algebra g, a Kac–Moody algebra $\hat{g}$ of level k contributes positively or negatively $k \dim g/(k+h)$ (h being the Coxeter number of g) to the total central charge, depending on whether it is part of the algebra or subalgebra, respectively, one obtains the famous sequence

$$c = 1 - \frac{6}{(k+2)(k+3)} \qquad k = 1, 2, 3, \ldots. \tag{11}$$

Including now the relevant $U(1)$-factors, we may also obtain the series (11) from a coset of two $\mathbf{g}|\tilde{\mathbf{g}}$-theories

$$A_1|A_1^{\otimes 2} \otimes A_{k-1}|A_1/A_k|A_1 \quad \Leftrightarrow \quad A_1|A_k/A_1|A_{k-1} \tag{12}$$

in the ultraviolet limit. We formally view $A_1|A_0$ and $A_0|A_1$ as unity $\mathbb{I}$ contributing 0 to the central charge. Relation (12) allows for various interpretations with regard to the realizations of several RG-flows. We note that both theories on the lhs do not contain any unstable particle. A flow between cosets parametrized by different k's may then be achieved in the so-called massless way as roaming trajectories in the spirit of [19]. On the other hand, the realizations in the form of the rhs of (12) constitute theories which contain unstable particles. Therefore, a flow between cosets related to different k's is achievable in a well-controllable fashion over the different energy scales of the unstable particles as observed in [4, 20–23] for the HSG-models. For vanishing resonance parameters σ_{ij} the system on the rhs of (12) leads to the same constant TBA-equations as found for the RSOS-models [24]. In addition, following the RG-flow of the scaling function of the TBA one observes that at the fixed points, the set of equations (6) is also obtained for finite values of the resonance parameters.

Of course, these coset realizations are not unique and one may, for instance, also obtain (11) from the quaternionic projective space HP^k [17] or use various exceptional Lie algebras to construct particular theories. This ambiguity allows for various other realizations in terms of different combinations of HSG-models.

2.2.2. Unitary N = 1 super conformal field theories. The series of $N = 1$ unitary minimal models $\mathcal{M}^{N=1}(k, k+1)$ has played an important role in the construction of certain string theories. It may be realized, for instance, by the cosets $SU(2)_k \otimes SU(2)_2/SU(2)_{k+2}$ or $SU(k+2)_2/SU(k)_2 \otimes SU(2)_2$ [17]. The corresponding series for the Virasoro central charge is

$$c = \frac{3}{2} - \frac{12}{(k+2)(k+4)} \qquad k = 1, 2, 3, \ldots. \tag{13}$$

Once again we may include the relevant $U(1)$-factors and also construct the $\mathcal{M}^{N=1}(k, k+1)$ models from several $\mathbf{g}|\tilde{\mathbf{g}}$-theories, for instance

$$A_{k-1}|A_1 \otimes A_1|A_1^{\otimes 3}/A_{k+1}|A_1 \quad \Leftrightarrow \quad A_1|A_{k+1}/A_1|A_{k-1} \otimes A_1|A_1. \tag{14}$$

In the ultraviolet limit they posses central charges of the form (13). Once again we note that there is a realization which involves unstable particles, i.e. the rhs of (14), and one which does not, i.e. the lhs of (14).

2.2.3. Unitary N = 2 super conformal field theories. The series of $N = 2$ unitary minimal models $\mathcal{M}^{N=2}(k, k+1)$ is omnipresent in string theory [25] (for a recent review see e.g. [26]). It may be realized by the cosets $SU(2)_k \otimes U(1)/U(1)$ or $SO(2k)_2/SU(k)_2$ with the corresponding series of the Virasoro central charge

$$c = \frac{3k}{2+k} \qquad k = 1, 2, 3, \ldots. \tag{15}$$

Including the relevant $U(1)$-factors, we construct from several $\mathbf{g}|\tilde{\mathbf{g}}$-theories the realizations

$$A_{k-1}|A_1 \otimes A_3|A_1 \quad \Leftrightarrow \quad A_1|D_{k+1}/A_1|A_{k-1}. \tag{16}$$

In the ultraviolet limit they also lead to (15). A further possibility, which we shall exploit in section 3.4, to obtain (15), is to use the coset $A_1|D_{k+2}/A_1|A_{k-1} \otimes A_1|A_1^{\otimes 2}$. Once again we note that there is a realization which involves unstable particles, i.e. the rhs of (16), and one which does not, i.e. the lhs of (16).

2.2.4. $G_k \otimes G_l/G_{k+l}$-cosets. The $G_k \otimes G_l/G_{k+l}$-cosets are the more general theories which encompass various models. For instance, taking $G = SU(2)$ and setting $l = 2$ or $l = k-2, k = 1$, one obtains the $\mathcal{M}^{N=1}(k, k+1)$ or $\mathcal{M}(k, k+1)$-models, respectively. Massless flows related to these models were investigated in [19]. Once again there exists a realization in terms of HSG-models,

$$A_{k-1}|G \otimes A_{l-1}|G \otimes A_1|A_1^{\otimes 2\ell}/A_{k+l-1}|G \tag{17}$$

such that we may also reproduce these flows by means of a variation of the energy scales of the unstable particles. Here ℓ is still the rank of the Lie algebra g. We will not perform a detailed investigation of these theories which go beyond the $\mathcal{M}^{N=1}(k, k+1)$ or $\mathcal{M}(k, k+1)$-models, but from the following analysis it will become apparent that the existence of realization (17) allows for an analogue treatment, when taking the non-unitary nature of many of these models into account.

3. RG-flow from q-deformed Virasoro characters

We now wish to introduce a mass scale. Recalling [27, 28] that the recurrence relations (6) may be solved by means of principally specialized Weyl characters, a natural conjecture is to suspect that a deformation of these expressions leads to a correct description of the massive theories in the sense of the full TBA-equations. Making this concrete seems a rather difficult task and we therefore construct a scaling function in a different way, but nonetheless in the spirit of the renormalization group ideas. Instead of using a different parametrization for the principally specialized Weyl characters, we deform the Virasoro characters (4) in a very natural way. As was already pointed out in the previous section, the saddle point analysis which leads to equations (6) and (7) does not depend on the fact whether the matrix M is constant or variable. We can exploit this by introducing mass scales in a rather suggestive fashion. Restricting ourselves to the large class of simply laced $\mathbf{g}|\tilde{\mathbf{g}}$-theories and cosets constructed from these theories as in section 2.2, we replace now the M-matrix by a q-deformed version

$$\left[M_{ab}^{ij}\right]_q := [K_{ab}]_q^{-1}[\tilde{K}_{ij}]_{\tilde{q}_{ij}} \tag{18}$$

with

$$[K_{ab}]_q := K_{ab}q = \alpha_a \cdot \alpha_b q = \alpha_a \cdot \alpha_b \exp(-mr/2) \tag{19}$$

$$[\tilde{K}_{ij}]_{\tilde{q}_{ij}} := 2\delta_{ij} - [\tilde{I}_{ij}]_{\tilde{q}_{ij}} = \tilde{\alpha}_i \cdot \tilde{\alpha}_j \tilde{q}_{ij} = \tilde{\alpha}_i \cdot \tilde{\alpha}_j \exp\left(-mr/2(1-\delta_{ij})\mathrm{e}^{|\sigma_{ij}|/2}\right). \tag{20}$$

Here α_i, $\tilde{\alpha}_i$ are the simple roots of $\mathbf{g}$, $\tilde{\mathbf{g}}$, respectively. In other words, we re-defined the usual scalar product between the simple roots or equivalently q-deformed the roots themselves. The bracket $[\,]_q$ is not to be confused with the usual notation of q-deformed integers. Q-deformations of a different nature have recently played an important role in the context of the formulation of consistent expressions for scattering matrices of affine Toda field theories related to non-simply laced Lie algebras [29]. For the case at hand the q-deformation is mainly inspired by the physics of the unstable particles. The natural mass scale of the unstable particle $m_{\tilde{c}} \sim mr/2\mathrm{e}^{|\sigma_{ij}|/2}$, with σ_{ij} playing the role of a resonance parameter and m of an overall mass scale, is introduced in $\tilde{K}$ in such a way that for $\sigma_{ij} \to \infty$, the Cartan matrix of $\tilde{\mathbf{g}}$ decouples according to the 'cutting rule' analysed in [23]. Note that for $mr/2\mathrm{e}^{\sigma_{ij}/2} \ll 1$ we have $[\tilde{K}_{ij}]_{\tilde{q}_{ij}} \approx \tilde{K}_{ij}$, such that the decoupling takes place at the same scale as in the massive models (see e.g. equation (51) in [20] and also [4, 23]). In addition, we would like the particles to be massless in the infrared. Recalling that the masses of the affine Toda field theories can be organized in the form of the Perron–Frobenius vector of the Cartan matrix, deformation (19) achieves this goal. In the limit $r \to 0$ we recover the usual Cartan matrix.

Of course, the deformations of types (19) and (20) are not unique and one could try to find different realizations in order to construct scaling functions. However, from the arguments just outlined they appear to be the most natural ones.

3.1. $\mathbf{g}|\tilde{\mathbf{g}}$ *-theories*

Equipped with matrices (19) and (20), the q-deformed version of (4) acquires the form

$$\chi(q, r, \vec{m}, \vec{\sigma}) = \sum_{\vec{k}=0}^{\infty} \frac{q^{\frac{1}{2}\vec{k}[M]_{\{r,\vec{m},\vec{\sigma}\}}\vec{k}^t + \vec{k}\cdot\vec{B}}}{(q)_{k_1}\ldots(q)_{k_n}}. \tag{21}$$

For simplicity we took here l' to be zero. We collect the $\tilde{\ell} - 1$ linearly independent resonance parameters in the vector $\vec{\sigma}$ and the ℓ independent mass scales in $\vec{m}$. The RG scaling parameter is denoted by r. To obtain the recurrence relations in a more symmetric way it is convenient to introduce the variables $x_a^i = \prod_{b=1}^{\ell}\big(Q_b^i\big)^{-K_{ab}}$. In terms of the q-deformed analogues to these variables, $\big[x_a^i\big]_q = \prod_{b=1}^{\ell}\big(Q_b^i\big)^{-[K_{ab}]_q}$, the saddle point analysis of (21) leads to

$$\prod_{b=1}^{\ell} Q_b^i(r, \vec{m}, \vec{\sigma})^{-[K_{ab}]_q} + \prod_{j=1}^{\tilde{\ell}} Q_a^j(r, \vec{m}, \vec{\sigma})^{-[\tilde{K}_{ij}]_{\tilde{q}_{ij}}} = 1 \tag{22}$$

together with the associated scaling function

$$c^{\mathbf{g}|\tilde{\mathbf{g}}}(r, \vec{m}, \vec{\sigma}) = \frac{6}{\pi^2}\sum_{a=1}^{\ell}\sum_{i=1}^{\tilde{\ell}} \mathcal{L}\left(\prod_{j=1}^{\tilde{\ell}} Q_a^j(r, \vec{m}, \vec{\sigma})^{-[\tilde{K}_{ij}]_{q_{ij}}}\right). \tag{23}$$

The recurrence relations (22) now play an analogous role to the TBA-equations. In order to make our main point, namely that (23) indeed constitutes a scaling function which reproduces the characteristic features of the theory, like the ones obtainable from the conventional TBA, the scaled version of the c-theorem or a semiclassical analysis, we have to establish that $c^{\mathbf{g}|\tilde{\mathbf{g}}}(r, \vec{m}, \vec{\sigma})$ indeed satisfies properties (i)–(iv) in the introduction.

Most straightforward to prove are the properties related to the extremal limits. Property (i) is easily established since by construction $c^{\mathbf{g}|\tilde{\mathbf{g}}}(0, \vec{m}, \vec{\sigma})$ is the ultraviolet Virasoro central charge. Property (iv) follows from the following argument: let us first assume in (22) that the Q_a^i are finite for $r \to \infty$. Then taking this limit leads to $1 + \big(Q_a^i\big)^{-2} = 1$, such that our initial assumption cannot hold and we deduce that $\lim_{r\to\infty} Q_a^i \sim \infty$. When we want to avoid that the scaling function (23) becomes complex we have to assume that the Q's are real. Additional support for this assumption will be provided below just based on the structure of (22) and a possible physical interpretation. Thus taking now $Q \in \mathbb{R}$ each term on the lhs of (22) has to be smaller than 1, such that we deduce for the infrared asymptotics of the first term

$$\lim_{r\to\infty} \mathrm{e}^{-mr/2} \sum_b K_{ab} \ln Q_b^i = 0. \tag{24}$$

Excluding the exotic case $\sum_b K_{ab} \ln Q_b^i = 0$, we demand the behaviour (24) for each term in the sum and conclude that the second term in (22) is zero such that with $\mathcal{L}(0) = 0$ we finally conclude that property (iv) holds.

The other properties are less straightforward to prove in complete generality and we will be content to establish them on the basis of explicit case-by-case examples.

3.2. $\mathbf{A}_1|\tilde{\mathbf{g}} \equiv \tilde{\mathbf{g}}_2$*-HSG*

The $\mathbf{A}_1|\tilde{\mathbf{g}}$-theories are good theories to start with, since they do not involve any stable particle fusing structure. In addition, several scaling functions have been obtained by a TBA analysis [20] and also from the scaled version of the c-theorem [4, 23], such that we already have data available to compare with. Equations (22) in this case simply become

$$Q^i(r, m, \vec{\sigma})^2 = Q^i(r, m, \vec{\sigma})^{2-2q} + \prod_{j=1}^{\tilde{\ell}} Q^j(r, m, \vec{\sigma})^{[\tilde{I}_{ij}]_{q_{ij}}}. \tag{25}$$

It is useful to treat the case $\tilde{\mathbf{g}} = \boldsymbol{A}_1$ separately, since it corresponds to the free fermion.

3.2.1. The free fermion. The free fermion is analytically solvable in several approaches and is therefore an ideal example to illustrate that the various scaling functions are quantitatively different but contain qualitatively the same information. Equation (25) in this case simply becomes $Q^2 = Q^{2-2q} + 1$. It is not possible to solve this relation analytically, but near the ultraviolet we may approximate $q \approx 1$ such that its solution becomes $Q \sim \sqrt{2}$ for $rm/2 \ll 1$, and therefore

$$c^{\mathbf{A}_1|\mathbf{A}_1}(rm) \sim \frac{6}{\pi^2}\mathcal{L}(1/2) = \frac{1}{2} \qquad \text{for} \quad rm/2 \ll 1. \tag{26}$$

We can compare this with the scaling function obtained as an exact solution of the full TBA analysis

$$c^{\mathrm{TBA}}(rm) = \frac{6rm}{\pi^2} \sum_{n=1}^{\infty} (-1)^n \frac{K_1(nrm)}{n} \sim \frac{1}{2} \qquad \text{for} \quad rm/2 \ll 1 \tag{27}$$

where K_1 is a modified Bessel function. The latter estimate follows from $K_1(rm) \sim 1/rm$ for $rm/2 \ll 1$ and the fact that $\mathcal{L}(-1) = -12/\pi^2$. This means that in the main region of interest these two functions coincide. It is also clear that for large rm both functions vanish.

In addition, we may compare with the scaling function obtained from the c-theorem

$$c^{c\text{-th}}(rm) = \frac{3}{2} \int_{rm}^{\infty} \mathrm{d}s\, s^3 \left(K_1(s)^2 - K_0(s)^2\right) \sim \frac{1}{2} \qquad \text{for} \quad rm/2 \ll 1 \tag{28}$$

which shows a similar behaviour. Note that despite the fact that we use rm in (26)–(28) the meaning of this parameter is different in each context. For our purposes it is simply a dimensionless variable.

Let us now establish property (ii) for this case. This illustrates at the same time the general procedure which works, in principle, for all other situations. Since we know that

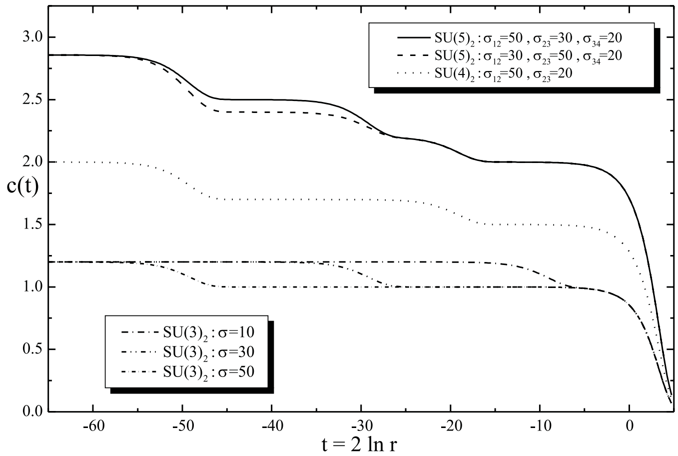


Figure 1. RG flow from q-deformed Virasoro characters.

$Q(r=0)=\sqrt{2}$ and $\lim_{r\to\infty} Q \to \infty$, we just have to establish that $Q(r)$ does not posses a minimum or maximum in order to establish its monotonic behaviour. We compute from (25) the derivative $Q' = q \ln Q/(2Q^{2q-1} - Q^{-1}(2-2q))$. Obviously, for finite values of Q, this only vanishes for $Q=1$, which is however not a solution of (25). Therefore Q does not have an extremum and property (ii) holds. Property (iii) holds trivially in this case.

3.2.2. $\tilde{\mathbf{g}} \neq \mathbf{A}_1$. For the other cases one may, in principle, proceed in a similar fashion, but already for the case $\mathbf{A}_1|\mathbf{A}_2$ the analysis becomes rather messy. For instance, computing the derivative in this case, we find that it only vanishes for $Q = \left(\frac{1}{2}\exp(mr/2(1-\exp(\sigma/2)+\sigma/2))\right)^{1/(2-2q-\tilde{q})}$. Substituting this back into (25) we find for a fixed value of σ a specific value of r such that the equation is satisfied. We may then compute the second derivative and establish that this value corresponds to a saddle point, which, in comparison with our numerical solution exhibited in figure 1, is indeed situated on the second plateau.

Since an analytic solution of (25) eluded our analysis so far, we will now resort to a numerical analysis. For this purpose we discretize the equation

$$Q^i_{(n+1)}(r,m,\vec{\sigma}) = \left(Q^i_{(n)}(r,m,\vec{\sigma})^{2-2\exp(-mr/2)} + \prod_{j=1}^{\tilde{\ell}} Q^j_{(n)}(r,m,\vec{\sigma})^{[\tilde{I}_{ij}]_{q_{ij}}} \right)^{1/2} \tag{29}$$

and solve it iteratively in the usual fashion. Assuming convergence of this procedure the value $n\to\infty$ is identified with the exact solution of the recurrence relations (25). We start with $r=0$ and set the initial value Q_0^j to be the analytically known (see section 4) solutions of the constant TBA-equations. Once we have achieved convergence for a particular value of r, we may increase this value by an amount δr and take always as a starting value the previous solution of (29). It turns out that this procedure is extremely fast convergent even when the particle number involved is very high. In comparison with the full TBA-equations, (29) are by far easier to solve since they do not involve the complication of a convolution and correspond technically at each value of r to a constant TBA-equation.

Figure 1 shows the numerical solution of (29) for various algebras and different choices of the relative order of magnitude of the resonance parameters. We reproduce precisely the same qualitative behaviour for the scaling function as obtained in the full TBA analysis [20]

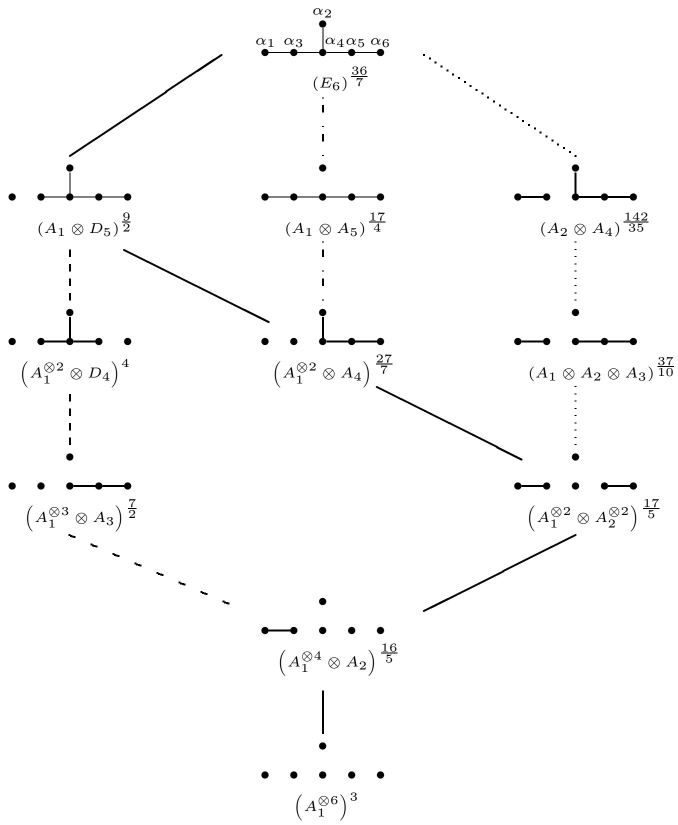


Figure 2. The decoupling of the $\mathbf{A}_1|\mathbf{E}_6$-theory.

and from the c-theorem [4, 23]. We recover all plateaux in the expected positions. In addition, we have the important property, as seen in figure 1 for the $SU(3)_2$-case, that a shift in σ by x may be compensated by a shift in t by the same amount.

3.3. $\mathbf{A}_1|\mathbf{E}_6 \equiv (E_6)_2$*-HSG*

The approach presented in this section even allows us to tackle more complicated algebras with relatively little effort, which in the full TBA analysis or the form factor approach constitutes a considerable computational problem. We illustrate this by considering the $\mathbf{A_1}|\mathbf{E_6}$-theory.

In figure 2 we present the decoupling of this theory and report the Virasoro central charges which are taken up along the flow as superscripts. In figure 3 we report the corresponding numerical results of (22) and (23) for this theory for various different choices of the relative order of magnitude of the resonance parameters. Our results precisely reproduce the central charges of figure 2.

3.4. $\mathbf{g}|\tilde{\mathbf{g}}$*-coset theories*

Recalling now from section 2.2 the various ways in which we can represent the unitary series, we may construct the flows between different cosets in a similar way as in the preceding

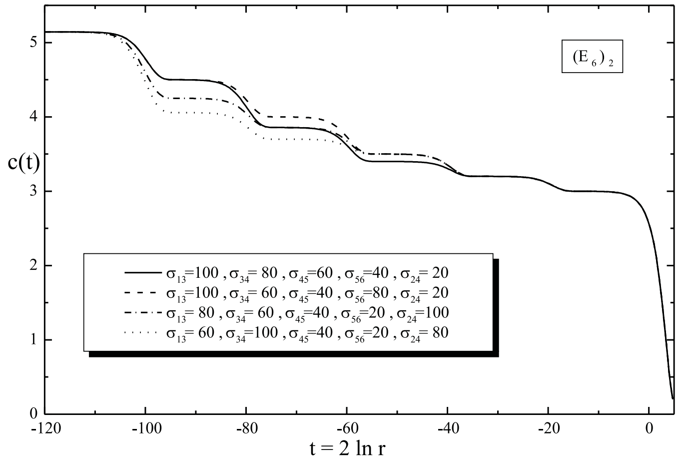


Figure 3. RG-flow from q-deformed Virasoro characters.

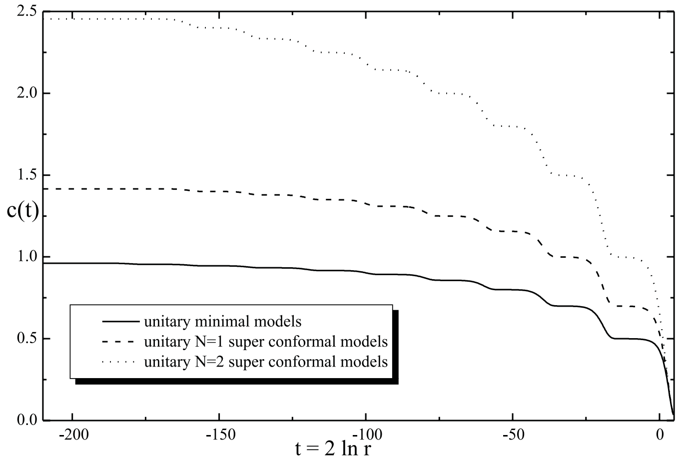


Figure 4. Internal RG-flow for the $N = 0, 1, 2$ unitary minimal models.

section for a single homogeneous sine-Gordon theory. Figure 4 exhibits the flow along the unitary series of the $N = 0, 1, 2$ superconformal minimal models.

From the realizations of the various cosets in terms of HSG-models it is also clear that we may produce flows between the different series as suggested in [6] by alternative means. By controlling the energy scale of the unstable particle, we obtain

$$\begin{aligned}
&\mathcal{M}^{N=2}(k, k+1) \equiv A_1|D_{k+2}/A_1|A_{k-1} \otimes A_1|A_1^{\otimes 2} \xrightarrow[\sigma_{k+1,k+2}\to\infty]{} \\
&\mathcal{M}^{N=1}(k, k+1) \equiv A_1|A_{k+1}/A_1|A_{k-1} \otimes A_1|A_1 \xrightarrow[\sigma_{k,k+1}\to\infty]{} \\
&\mathcal{M}(k, k+1) \equiv A_1|A_k/A_1|A_{k-1}.
\end{aligned}$$

Our numerical results which reproduce these flows are presented in figure 5. It is this type of flow which in [6] was realized as the so-called 'Bailey flow'.

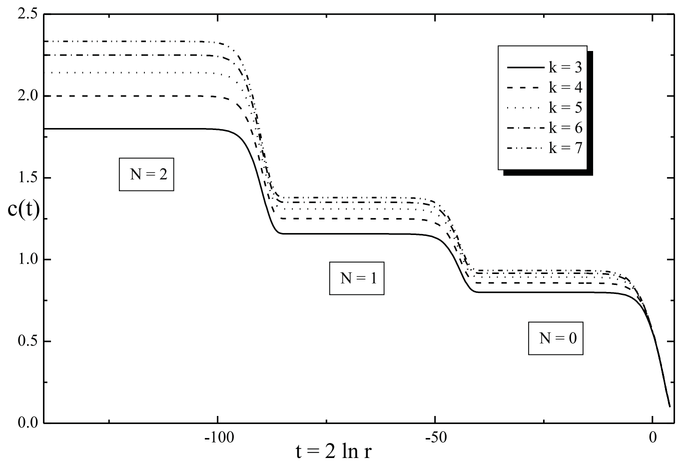


Figure 5. RG-flow between $N = 0, 1, 2$ unitary minimal models.

4. The fixed point solutions

As we have mentioned, we do not have a general solution of (22) so far for the entire range of r, but at each fixed point such expressions may be found. In [27, 28] it was noted that the recurrence relations (6) admit closed analytical solutions in terms of some very distinct mathematical objects, namely principally specialized Weyl characters. Since the proofs of these identities are very often missing or only indicated in the literature, we find it instructive to present various transparent proofs in this section. In addition, we present numerous new solutions for theories treated before and for some hitherto not considered at all. We start by assembling several properties of the characters which we utilize later to solve the recurrence relations (6) or equivalently (22) in the range of r characterized by property (iii) in the introduction.

4.1. Properties of Weyl characters

The characters for an irreducible representation D^λ of a simple Lie algebra $\mathbf{g}$ with rank ℓ are well known to be expressible in terms of the famous Weyl character formula, see e.g. [30],

$$\hat{\chi}_\lambda\,[A(t)] = \mathrm{Tr}[D^\lambda(g)] = \mathrm{Tr}\big[D^\lambda\big(\mathrm{e}^{\mathrm{i}A(t)}\big)\big] = \frac{\sum_{\omega\in\mathcal{W}}(\det\omega)\mathrm{e}^{\mathrm{i}\omega(\lambda+\rho)\cdot t}}{\sum_{\omega\in\mathcal{W}}(\det\omega)\mathrm{e}^{\mathrm{i}\omega(\rho)\cdot t}}. \tag{30}$$

Here g is an element of the Lie group, $A(t) \in \mathbf{g}$,[2] $\mathcal{W}$ denotes the Weyl group, λ is an arbitrary weight and $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha = \sum_{i=1}^{\ell}\lambda_i$ is the Weyl vector with λ_i denoting the fundamental weights. It is also known that, when computing the character for the principal $SU(2)$ subalgebra of $\mathbf{g}$, see [31], the character (30) factorizes into the form

$$\hat{\chi}_\lambda\,[\tau\rho\cdot H] = \chi_\lambda(\tau) = \prod_{\alpha\in\Delta_+}\frac{\sin\,(\alpha\cdot(\lambda+\rho)\pi\tau)}{\sin\,(\alpha\cdot\rho\pi\tau)} \tag{31}$$

where Δ_+ is the set of positive roots. We refer to the particular expression $\chi_\lambda(\tau)$ of the Weyl character as the *principally specialized Weyl character* (PSW-character). When considering λ

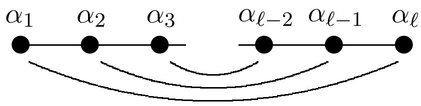
[2] Due to fact that the elements of the group may be written as $g = fhf^{-1}$ together with the cyclic property of the trace, one can take $A(t) = t\cdot H$ to be an element of the Cartan subagebra.

to be a fundamental weight λ_i, it is useful to employ the conventions $\chi_{\lambda_0} = \chi_{\lambda_{\ell+1}} = 1$ and set $\chi_{\lambda_{-n}} = 0$ for a positive integer n. When τ approaches 0, we obtain the well-known formula for the dimension of the particular representation of the weight λ:

$$\dim \lambda = \prod_{\alpha \in \Delta_+} \frac{\alpha \cdot (\lambda + \rho)}{\alpha \cdot \rho}. \tag{32}$$

We now wish to establish various properties for the character $\chi_\lambda(\tau)$. It appears difficult to carry out these studies on the generic expression (31) and we shall therefore resort to a case-by-case analysis. Denoting by $\varepsilon_1, \ldots, \varepsilon_n$ the standard orthonormal basis of $\mathbb{R}^n$ with $\varepsilon_i \cdot \varepsilon_j = \delta_{ij}$, it is well-known that it is possible to represent the entire root system as vectors on a suitably chosen lattice in $\mathbb{R}^n$ with one (simply laced) or two (non-simply laced) prescribed lengths. We adopt the conventions of Bourbaki [32], which resulted historically from an investigation of the adjoint representation of simple Lie algebras, which is the reason why they do not always appear entirely obvious.

4.1.1. A_ℓ



We represent the roots of A_ℓ in $\mathbb{R}^{\ell+1}$. According to [32] all positive roots are given by

$$\varepsilon_i - \varepsilon_j = \alpha \in \Delta_+ \qquad \text{for} \quad 1 \leqslant i < j \leqslant \ell + 1. \tag{33}$$

The fundamental weights and the Weyl vector are realized as

$$\lambda_k = \sum_{i=1}^{k} \varepsilon_i - \frac{k}{\ell+1} \sum_{i=1}^{\ell+1} \varepsilon_i \qquad \text{and} \qquad \rho = \sum_{i=1}^{\ell+1} (\ell/2 + 1 - i)\varepsilon_i. \tag{34}$$

Equipped with these quantities we can evaluate (31) and obtain more explicit formulae

$$\chi_{a\lambda_k}(\tau) = \prod_{1 \leqslant i < j \leqslant \ell+1} \frac{\sin[(\varepsilon_i - \varepsilon_j) \cdot (a\lambda_k + \rho)\pi\tau]}{\sin[(\varepsilon_i - \varepsilon_j) \cdot \rho\pi\tau]} = \prod_{i=1}^{k} \prod_{j=k}^{\ell} \frac{\sin[(a + 1 + j - i)\pi\tau]}{\sin[(1 + j - i)\pi\tau]}. \tag{35}$$

The last expression in (35) is best suited to establish various properties of the A_ℓ-related characters,

$$\chi_{a\lambda_k}(\tau) = \chi_{a\lambda_k}(\tau + 2) \tag{36}$$

$$\chi_{a\lambda_k}(\tau) = \chi_{a\lambda_{\ell+1-k}}(\tau) \tag{37}$$

$$\chi_{(a+1)\lambda_k}(\tau) = \chi_{a\lambda_k}(\tau) \prod_{j=1}^{k} \frac{\sin[(a + \ell + 2 - j)\pi\tau]}{\sin[(a + k + 1 - j)\pi\tau]} \tag{38}$$

$$\chi_{a\lambda_{k+1}}(\tau) = \chi_{a\lambda_k}(\tau) \prod_{j=1+k}^{\ell} \frac{\sin[(a + j)\pi\tau]}{\sin[j\pi\tau]} \prod_{i=1}^{k} \frac{\sin[(\ell + 1 - j)\pi\tau]}{\sin[(a + \ell + 1 - j)\pi\tau]} \tag{39}$$

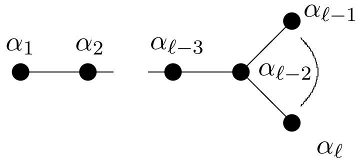
$$\chi_{a\lambda_k}(\tau)\chi_{a\lambda_k}(\tau) = \chi_{(a+1)\lambda_k}(\tau)\chi_{(a-1)\lambda_k}(\tau) + \chi_{a\lambda_{k+1}}(\tau)\chi_{a\lambda_{k-1}}(\tau). \tag{40}$$

Here (36) is obvious and (37)–(39) follow from simple shifts in (35). With the help of (38) and (39) we can verify (40). Note that (36)–(40) hold for generic values of τ. We now also want to identify $\chi_{a\lambda_k}$ and $\chi_{(\tilde{l}+1-a)\lambda_k}$ for some integer $\tilde{l}$. This is, however, not true for generic values of τ. Expressing $\chi_{a\lambda_k}$ and $\chi_{(\tilde{l}+1-a)\lambda_k}$ in the form (35) and denoting the variables over which the products are taken in the former by i, j and the latter by i', j', the two characters obviously coincide if $(a+j-i)\tau = 1 + (a-\tilde{l}-1-j'+i')\tau$. From the available values of i, j, i', j' the combination $j + j' - i - i' = \ell + 1$ constitutes a consistent solution of this equation such that we have

$$\chi_{a\lambda_k}\left(\tau = \tfrac{1}{\ell+\tilde{\ell}+2}\right) = \chi_{(\tilde{l}+1-a)\lambda_k}\left(\tau = \tfrac{1}{\ell+\tilde{\ell}+2}\right). \tag{41}$$

This means it is the symmetry of the Dynkin diagram which fixes the value of τ.

4.1.2. D_ℓ



We represent the roots of D_ℓ in $\mathbb{R}^\ell$. According to [32] all positive roots are expressible as

$$\varepsilon_i \pm \varepsilon_j = \alpha \in \Delta_+ \qquad \text{for} \quad 1 \leqslant i < j \leqslant \ell. \tag{42}$$

The fundamental weights are given by

$$\lambda_{\ell-1} = \sum_{i=1}^{\ell-1} \frac{\varepsilon_i - \varepsilon_\ell}{2} \qquad \lambda_\ell = \frac{1}{2}\sum_{i=1}^{\ell} \varepsilon_i \qquad \lambda_k = \sum_{i=1}^{k} \varepsilon_i \qquad \text{for} \quad 1 \leqslant k \leqslant \ell - 2 \tag{43}$$

such that the Weyl vector reads

$$\rho = \sum_{i=1}^{\ell-1} (\ell - i)\varepsilon_i. \tag{44}$$

Substituting these quantities into (31) yields

$$\chi_{a\lambda_k}(\tau) = \prod_{1\leqslant i<j\leqslant \ell} \frac{\sin[(\varepsilon_i - \varepsilon_j)\cdot(a\lambda_k + \rho)\pi\tau]}{\sin[(\varepsilon_i - \varepsilon_j)\cdot\rho\pi\tau]} \frac{\sin[(\varepsilon_i + \varepsilon_j)\cdot(a\lambda_k + \rho)\pi\tau]}{\sin[(\varepsilon_i + \varepsilon_j)\cdot\rho\pi\tau]} \tag{45}$$

from which we derive

$$\begin{aligned}\chi_{a\lambda_k}(\tau) = {} & \prod_{1\leqslant i<j\leqslant k} \frac{\sin[(2a+2\ell-i-j)\pi\tau]}{\sin[(2\ell-i-j)\pi\tau]} \prod_{i=1}^{k}\prod_{j=k+1}^{\ell} \frac{\sin[(a+j-i)\pi\tau]}{\sin[(j-i)\pi\tau]} \\ & \times \frac{\sin[(2\ell+a-j-i)\pi\tau]}{\sin[(2\ell-j-i)\pi\tau]} \qquad 1 \leqslant k \leqslant \ell-2\end{aligned} \tag{46}$$

$$\chi_{a\lambda_\ell}(\tau) = \chi_{a\lambda_{\ell-1}}(\tau) = \prod_{1\leqslant i<j\leqslant \ell} \frac{\sin[(2\ell+a-i-j)\pi\tau]}{\sin[(2\ell-i-j)\pi\tau]}. \tag{47}$$

From (46) and (47) we can now deduce various properties of the D_ℓ-related characters,

$$\chi_{a\lambda_k}(\tau) = \chi_{a\lambda_k}(\tau+2) \tag{48}$$

$$\chi_{a\lambda_\ell}(\tau) = \chi_{a\lambda_{\ell-1}}(\tau) \tag{49}$$

$$\chi_{a\lambda_1}(\tau') = \sum_{k=0}^{\infty} (-1)^k \chi_{\lambda_{a-2k}}(\tau') \qquad a \leqslant \ell - 2 \tag{50}$$

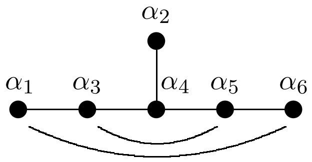
$$\chi_{\lambda_{n+1}}(\tau') = \chi_{(n+1)\lambda_1}(\tau') + \chi_{(n-1)\lambda_1}(\tau') \tag{51}$$

$$\chi_{\lambda_1}(\tau')\chi_{\lambda_1}(\tau') = 2\chi_{\lambda_2}(\tau') \tag{52}$$

$$\chi_{\lambda_\ell}(\tau')\chi_{\lambda_\ell}(\tau') = 2\sum_{k=0}^{\infty} \chi_{\lambda_{\ell-2-4k}}(\tau'). \tag{53}$$

Here we have set $\tau' = 1/(4\ell - 4)$.

4.1.3. E_6



Following still [32] the roots and weights of E_6 may be represented in $\mathbb{R}^8$, where we label the roots as depicted in the preceding Dynkin diagram. Since these expressions are rather cumbersome, we refer the reader to the literature and report here only the final expressions for the characters. Noting that all characters are of the general form $\prod_{1\leqslant x<h}\sin(\pi\tau(a+x))/\sin(\pi\tau x)$, with h being the Coxeter number, it is convenient to use the following notation:

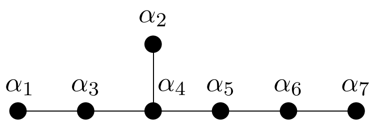
$$\left\{a_{1,1}^{x_{1,1}},\ldots,a_{1,b_1}^{x_{1,b_1}};\ldots;a_{i,1}^{x_{i,1}},a_{i,2}^{x_{i,2}},\ldots,a_{i,b_i}^{x_{i,b_i}},\ldots\right\}:=\prod_{i=1}^{h-1}\prod_{j=1}^{b_i}\left(\frac{\sin\pi\tau(a_{i,j}+i)}{\sin\pi\tau i}\right)^{x_{i,j}}. \tag{54}$$

Note that all expressions we find have at least one $x_{i,j}\neq 0$ for each $i\in[1,h-1]$. We compute

$$\chi_{a\lambda_1}=\{a;a;a;a^2;a^2;a^2;a^2;a^2;a;a;a\} \tag{55}$$
$$\chi_{a\lambda_2}=\{a;a;a^2;a^3;a^3;a^3;a^3;a^2;a;a;2a\} \tag{56}$$
$$\chi_{a\lambda_3}=\{a;a^2;a^3;a^4;a^4;a^3;a^2,2a;a,2a;2a;2a;2a\} \tag{57}$$
$$\chi_{a\lambda_4}=\{a;a^3;a^5;a^5;a^3,2a;a,2a^2;2a^3;2a^2;2a;3a;3a\}. \tag{58}$$

Here and in the following we suppress the τ-dependence of χ, i.e. we read $\chi_{a\lambda_i}=\chi_{a\lambda_i}(\tau)$.

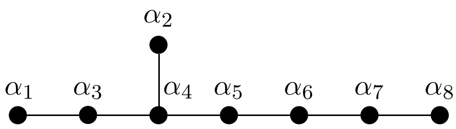
4.1.4. E_7



Our conventions for naming the roots are the same as in [32] according to which we represent the roots of E_7 in $\mathbb{R}^8$. We then compute

$$\chi_{a\lambda_1}=\{a;a;a;a^2;a^2;a^3;a^3;a^3;a^3;a^3;a^3;a^2;a^2;a;a;a;2a\} \tag{59}$$
$$\chi_{a\lambda_2}=\{a;a;a^2;a^3;a^4;a^4;a^5;a^4;a^4;a^3;a^2,2a;a,2a;a,2a;2a;2a;2a;2a\} \tag{60}$$
$$\chi_{a\lambda_3}=\{a;a^2;a^3;a^4;a^5;a^5;a^4,2a;a^3,2a;a^2,2a^2;a,2a^2;2a^3;2a^2;2a^2;2a;2a;3a;3a\} \tag{61}$$
$$\chi_{a\lambda_4}=\{a;a^3;a^5;a^6;a^5,2a;a^3,2a^2;a,2a^4;2a^4;2a^4;2a^2,3a;2a,3a^2;3a^2;3a^2;3a;4a;4a;4a\} \tag{62}$$
$$\chi_{a\lambda_5}=\{a;a^2;a^4;a^5;a^6;a^5;a^4,2a;a^2,2a^2;a,2a^3;2a^3;2a^3;2a^2;2a,3a;3a;3a;3a;3a\} \tag{63}$$
$$\chi_{a\lambda_6}=\{a;a^2;a^2;a^3;a^4;a^4;a^4;a^4;a^3,2a;a^2,2a;a^2,2a;a,2a;2a^2;2a;2a;2a;2a\} \tag{64}$$
$$\chi_{a\lambda_7}=\{a;a;a;a;a^2;a^2;a^2;a^2;a^3;a^2;a^2;a^2;a^2;a;a;a;a\}. \tag{65}$$

4.1.5. E_8



Our conventions for naming the roots are as in [32] according to which we represent the roots of E_8 in $\mathbb{R}^8$. We compute

$$\chi_{a\lambda_1} = \{a; a; a; a^2; a^2; a^3; a^4; a^4; a^4; a^5; a^5; a^5; a^5; a^4; a^4; a^4; a^3, 2a; a^2, 2a; a^2, 2a; a, 2a; a, 2a; a, 2a; 2a^2; 2a; 2a; 2a; 2a; 2a; 2a\} \tag{66}$$

$$\chi_{a\lambda_2} = \{a; a; a^2; a^3; a^4; a^5; a^6; a^6; a^6; a^6; a^5, 2a; a^4, 2a; a^3, 2a^2; a^2, 2a^2; a, 2a^3; a, 2a^3; 2a^4; 2a^3; 2a^3; 2a^2; 2a^2, 2a; 3a; 2a; 3a; 3a; 3a; 3a; 3a; 3a; 3a\} \tag{67}$$

$$\chi_{a\lambda_3} = \{a; a^2; a^3; a^4; a^5; a^6; a^6, 2a; a^5, 2a; a^4, 2a^2; a^3, 2a^3; a^2, 2a^4; a, 2a^4; 2a^5; 2a^4; 2a^4; 2a^3, 3a; 2a^2, 3a; 2a, 3a^2; 2a, 3a^2; 3a^2; 3a^2; 3a^2; 3a, 4a; 4a; 4a; 4a; 4a; 4a; 4a\} \tag{68}$$

$$\chi_{a\lambda_4} = \{a; a^3; a^5; a^6; a^6, 2a; a^5, 2a^2; a^3, 2a^4; a, 2a^5; 2a^6; 2a^5, 3a; 2a^4, 3a^2; 2a^2, 3a^3; 2a, 3a^4; 3a^4; 3a^3, 4a; 3a^2, 4a^2; 3a, 4a^3; 4a^3; 4a^3; 4a^2; 4a, 5a; 5a^2; 5a^2; 5a; 6a; 6a; 6a; 6a; 6a\} \tag{69}$$

$$\chi_{a\lambda_5} = \{a; a^2; a^4; a^6; a^7; a^7; a^6, 2a; a^4, 2a^2; a^2, 2a^4; a, 2a^5; 2a^6; 2a^5; 2a^4, 3a; 2a^2, 3a^2; 2a^2, 3a^3; 3a^4; 3a^4; 3a^3; 3a^2, 4a; 3a, 4a; 4a^2; 4a^2; 4a^2; 4a; 5a; 5a; 5a; 5a\} \tag{70}$$

$$\chi_{a\lambda_6} = \{a; a^2; a^3; a^4; a^5; a^6; a^6; a^6; a^5, 2a; a^4, 2a^2; a^3, 2a^3; a^2, 2a^3; a, 2a^4; 2a^4; 2a^4; 2a^3, 3a; 2a^3, 3a; 2a^2, 3a; 2a, 3a^2; 3a^2; 3a^2; 3a^2; 3a^2; 3a; 3a; 3a; 4a; 4a, ; 4a\} \tag{71}$$

$$\chi_{a\lambda_7} = \{a; a^2; a^2; a^2; a^3; a^4; a^4; a^4; a^5; a^5; a^4, 2a; a^4, 2a; a^4, 2a; a^3, 2a; a^2, 2a^2; a^2, 2a^2; a^2, 2a^2; a, 2a^2; 2a^3; 2a^2; 2a^2; 2a^2; 2a^2; 2a; 2a; 2a; 2a; 3a; 3a\} \tag{72}$$

$$\chi_{a\lambda_8} = \{a; a; a; a; a; a^2; a^2; a^2; a^2; a^3; a^3; a^3; a^3; a^3; a^3; a^3; a^3; a^3; a^3; a^3; a^2; a^2; a^2; a^2; a; a; a; a; a; a\} \tag{73}$$

4.2. Solution for the $\mathbf{g}|\tilde{\mathbf{g}}$*-theories*

As already indicated in section 3, when introducing the variables $x_a^i = \prod_{b=1}^{\ell} \left(Q_b^i\right)^{-K_{ab}}$ the constant TBA-equations (6), or equivalently (22) at certain fixed points, acquire the more symmetric form

$$\left(Q_a^i\right)^2 = \prod_{b=1}^{\ell} \left(Q_b^i\right)^{I_{ab}} + \prod_{j=1}^{\tilde{\ell}} \left(Q_a^j\right)^{\tilde{I}_{ij}}. \tag{74}$$

It is convenient to take here $Q_a^0 = Q_0^i = 1$. We will now identify the Q's with various combinations of PSW-characters (74) either of the algebra $\mathbf{g}$ or $\tilde{\mathbf{g}}$ such that the relations (74) are solved. One should note here that in (74) the two algebras are on the same footing, despite the fact that on the level of the scattering matrix, i.e. the data which enter the Virasoro characters (4) and in (23), they play quite distinct roles. We always choose

$$\tau = \frac{1}{h + \tilde{h}} \tag{75}$$

in (31), with $h, \tilde{h}$ being the Coxeter numbers of $\mathbf{g}, \tilde{\mathbf{g}}$, respectively. It will be sufficient to concentrate on the $\mathbf{g}|\tilde{\mathbf{g}}$-theories, since the coset models reported in section 3.4 may be constructed simply by means of a system of type (6). Having solved (74) we also compute the (effective) central charge according to (23). In many cases this can be done analytically by reducing the expression to some well known (see e.g. [11]) numerical relations for the Rogers dilogarithm, such as $\mathcal{L}(1/2) = \pi^2/12$, $\mathcal{L}((\sqrt{5}-1)/2) = \pi^2/10$, etc. by a successive application of the five-term relation

$$\mathcal{L}(x) + \mathcal{L}(y) = \mathcal{L}(xy) + \mathcal{L}\left(\frac{x(1-y)}{1-xy}\right) + \mathcal{L}\left(\frac{y(1-x)}{1-xy}\right). \tag{76}$$

In several cases we do not attempt to be entirely rigorous and only verify the relations numerically. Especially when a generic rank is involved we only compute a large part of the beginning of the sequence and do not attempt to perform inductive proofs.

We proceed case by case.

4.2.1. $\mathbf{A}_\ell|\mathbf{A}_{\tilde{\ell}}$. In this case the recurrence relations (74) are explicitly

$$\left(Q_a^k\right)^2 = Q_{a+1}^k Q_{a-1}^k + Q_a^{k+1} Q_a^{k-1} \tag{77}$$

for $1 \leqslant a \leqslant \ell, 1 \leqslant k \leqslant \tilde{\ell}$. As was first pointed out in [27], by identifying the Q's with PSW-characters these relations may be solved explicitly. We may use either of the characters $\chi, \tilde{\chi}$ of $\mathbf{A}_\ell, \mathbf{A}_{\tilde{\ell}}$, respectively, with $\tau = 1/(\ell + \tilde{\ell} + 2)$,

$$Q_a^k = \chi_{k\lambda_a}(\tau) = \tilde{\chi}_{a\lambda_k}(\tau). \tag{78}$$

This follows now immediately by noting that (77) coincides precisely with equation (40). Using these solutions, the central charges according to (23) turn out to be

$$c = \frac{6}{\pi^2} \sum_{a=1}^{\ell} \sum_{k=1}^{\tilde{\ell}} \mathcal{L}\left(\frac{\tilde{\chi}_{a\lambda_{k-1}}(\tau)\tilde{\chi}_{a\lambda_{k+1}}(\tau)}{\tilde{\chi}_{a\lambda_k}(\tau)^2}\right) = \frac{\ell\tilde{\ell}(\tilde{\ell}+1)}{\ell + \tilde{\ell} + 2}. \tag{79}$$

4.2.2. $\mathbf{A}_1|\tilde{\mathbf{g}}$*-theories.* For the reasons mentioned in the previous section, these particular HSG-models are interesting to investigate. Exploiting the symmetry in equations (74), they may be solved by appealing to the solutions which correspond to those of minimal affine Toda field theories, i.e. $\mathbf{g}|\mathbf{A_1}$. These solutions in terms of the PSW-characters of $\mathbf{g}$ may be extracted from the general formulae provided in [27, 28]. The corresponding values were also stated thereafter in the first reference in [14] without proof. We demonstrate that alternatively one may simply use combinations of the characters of $\mathbf{A}_1$,

$$\chi_{k\lambda}(\tau) = \frac{\sin(\pi(1+k)\tau)}{\sin(\pi\tau)} \tag{80}$$

in order to solve the recurrence relations.

$\mathbf{A}_1|\mathbf{A}_{\tilde{\ell}}$. As a special case of (78) we obtain

$$Q^i = \tilde{\chi}_{\lambda_i}(1/(\tilde{\ell}+3)) = \chi_{i\lambda}(1/(\tilde{\ell}+3)). \tag{81}$$

Translating to the x-variables we recover the values quoted in [14]. The particularization of (23) yields the central charges

$$c = \frac{6}{\pi^2}\sum_{k=1}^{\tilde{\ell}} \mathcal{L}(1 - \tilde{\chi}_{\lambda_k}(1/(\tilde{\ell}+3))) = \frac{\tilde{\ell}(\tilde{\ell}+1)}{\tilde{\ell}+3}. \tag{82}$$

$\mathbf{A}_1|\mathbf{D}_{\tilde{\ell}}$. We may express the solutions for this case either in terms of the $\mathbf{D}_{\tilde{\ell}}$- or the $\mathbf{A}_1$-PSW-characters. Taking $\tau = 1/2\tilde{\ell}$, we obtain

$$Q^i = \sum_{k=1}^{\frac{i+1}{2}} \tilde{\chi}_{\lambda_{2k-1}}(\tau) = (i+1)\chi_{\lambda_{2\tilde{\ell}-2}}(\tau) = i+1 \qquad i \text{ odd}, i \leqslant \tilde{\ell}-2 \tag{83}$$

$$Q^i = 1 + \sum_{k=1}^{\frac{i}{2}} \tilde{\chi}_{\lambda_{2k}}(\tau) = (i+1)\chi_{\lambda_{2\tilde{\ell}-2}}(\tau) = i+1 \qquad i \text{ even}, i \leqslant \tilde{\ell}-2 \tag{84}$$

$$Q^{\tilde{\ell}-1} = \tilde{\chi}_{\lambda_{\tilde{\ell}-1}}(\tau) = Q^{\tilde{\ell}} = \tilde{\chi}_{\lambda_{\tilde{\ell}}}(\tau) = \sqrt{\tilde{\ell}} = \prod_{k=1}^{\tilde{\ell}-1} \chi_{(2k-1)\lambda}/\chi_{(\tilde{\ell}+k-1)\lambda}. \tag{85}$$

From the explicit expressions in section 4.1.2, it follows that $\tilde{\chi}_{\lambda k}(\tau) = 2$ for $k \leqslant \tilde{\ell}-2$ and the last relation in (85). Therefore, we may trivially evaluate the sums in (83) and (84), whose result we can employ to convince ourselves that (74) is indeed satisfied. Once again translating to the x-variables yields the values quoted in [14]. According to (23) the central charges are then computed as

$$c = \frac{6}{\pi^2}\left(\sum_{k=1}^{\tilde{\ell}-3} \mathcal{L}\left(\frac{k(k+2)}{(k+1)^2}\right) + \mathcal{L}\left(\frac{\tilde{\ell}(\tilde{\ell}-2)}{(\tilde{\ell}-1)^2}\right) + 2\mathcal{L}(1-\tilde{\ell}^{-1})\right) = \tilde{\ell}-1. \tag{86}$$

$\mathbf{A}_1|\mathbf{E}_6$. Using the conventions of section 4.1.3, the recurrence relations (74) in this case read

$$(Q^1)^2 = 1 + Q^3 \qquad (Q^2)^2 = 1 + Q^4 \qquad (Q^3)^2 = 1 + Q^4 Q^1 \qquad (Q^4)^2 = 1 + Q^2 (Q^3)^2 \tag{87}$$

where we have already exploited $Q^1 = Q^6$, $Q^3 = Q^5$, which is a consequence of the symmetry of the Dynkin diagram. For $a = 1$ and $\tau = 1/14$, expressions (55)–(58) for the E_6-characters reduce to

$$\tilde{\chi}_{\lambda_1} = \left(2\sin\frac{\pi}{14}\right)^{-1} \qquad \tilde{\chi}_{\lambda_2} = \tilde{\chi}_{\lambda_3} = 2\cos\frac{\pi}{7} \qquad \tilde{\chi}_{\lambda_4} = 0 \tag{88}$$

such that we can identify them with combinations of A_1-characters and *vice versa*

$$\tilde{\chi}_{\lambda_1} = 1 + \chi_{4\lambda} - \chi_{2\lambda} \qquad \tilde{\chi}_{\lambda_2} = \tilde{\chi}_{\lambda_3} = \chi_{2\lambda} - 1. \tag{89}$$

With these simple expressions for the characters, we may easily check that the expressions

$$Q^1 = 1 + \chi_{4\lambda} - \chi_{2\lambda} \qquad Q^2 = \chi_{2\lambda} \qquad Q^3 = \chi_{4\lambda} \qquad Q^4 = \chi_{4\lambda} + \chi_{2\lambda} \tag{90}$$

indeed satisfy the relations (87). Of course, with the help of (89) it is also possible to express the Q's in terms of the $\tilde{\chi}$'s instead of the χ's. Then making use of the symmetry between the two algebras in (74) and translating to the x-variables we recover the numerical values quoted in [14]. Assembling this, the central charge according to (23) is computed as

$$c = \frac{6}{\pi^2}\left(2\mathcal{L}\left(\frac{Q^3}{(Q^1)^2}\right) + \mathcal{L}\left(\frac{Q^4}{(Q^2)^2}\right) + 2\mathcal{L}\left(\frac{Q^1 Q^4}{(Q^3)^2}\right) + \mathcal{L}\left(\frac{(Q^3)^2 Q^2}{(Q^4)^2}\right)\right) = \frac{36}{7}. \tag{91}$$

$\mathbf{A}_1|\mathbf{E_7}$. With the conventions of section 4.1.4 the recurrence relations (74) in this case read

$$(Q^1)^2 = 1 + Q^3 \qquad (Q^2)^2 = 1 + Q^4 \qquad (Q^3)^2 = 1 + Q^4 Q^1 \qquad (Q^4)^2 = 1 + Q^3 Q^5 Q^2 \tag{92}$$

$$(Q^5)^2 = 1 + Q^4 Q^6 \qquad (Q^6)^2 = 1 + Q^7 Q^5 \qquad (Q^7)^2 = 1 + Q^6. \tag{93}$$

For $a = 1$ and $\tau = 1/20$, the expressions (59)–(65) for the E_7-characters simplify to

$$\tilde{\chi}_{\lambda_1} = \tilde{\chi}_{\lambda_6} = \frac{\sin\frac{3\pi}{5}}{\sin\frac{\pi}{5}} \qquad \tilde{\chi}_{\lambda_2} = \sqrt{2} \qquad \tilde{\chi}_{\lambda_3} = \tilde{\chi}_{\lambda_4} = \tilde{\chi}_{\lambda_5} = 0 \qquad \tilde{\chi}_{\lambda_7} = \frac{\sqrt{2}}{4\sin\frac{\pi}{20}\sin\frac{9\pi}{20}} \tag{94}$$

such that by recalling (80) we can identify them with combinations of A_1-characters and *vice versa*

$$\tilde{\chi}_{\lambda_1} = \tilde{\chi}_{\lambda_6} = \chi_{4\lambda} - \chi_{2\lambda} \qquad \tilde{\chi}_{\lambda_2} = \chi_{5\lambda} - \chi_{3\lambda} \qquad \tilde{\chi}_{\lambda_7} = \chi_{9\lambda} + \chi_{\lambda} - \chi_{7\lambda}. \tag{95}$$

With these simple expressions for the characters, we may once again verify after exploiting the symmetry of (74) or by direct analysis with the $\mathbf{A}_1$-characters that the expressions proposed in [28],

$$Q^1 = 1 + \tilde{\chi}_{\lambda_1} \qquad Q^2 = \tilde{\chi}_{\lambda_7} + \tilde{\chi}_{\lambda_2} \qquad Q^3 = 1 + 3\tilde{\chi}_{\lambda_1} \qquad Q^4 = 3 + 6\tilde{\chi}_{\lambda_1} \tag{96}$$

$$Q^5 = 2\tilde{\chi}_{\lambda_7} + 2\tilde{\chi}_{\lambda_2} \qquad Q^6 = 1 + 2\tilde{\chi}_{\lambda_1} \qquad Q^7 = \tilde{\chi}_{\lambda_7} \tag{97}$$

indeed satisfy (92)–(93)[3]. Renaming our roots and translating to the x-variables, we recover the numerical values quoted in [14]. The central charge (23) in this case is

$$c = \frac{6}{\pi^2}\left(\mathcal{L}\left(\frac{3\sqrt{5}-5}{2}\right) + \mathcal{L}\left(\frac{3\sqrt{5}-3)}{4}\right) + \mathcal{L}\left(\frac{3\sqrt{5}+3)}{10}\right) + \mathcal{L}\left(\frac{4\sqrt{5}}{9}\right)\right.$$
$$\left. + \mathcal{L}\left(\frac{3(3+\sqrt{5})}{16}\right) + \mathcal{L}\left(\frac{1+\sqrt{5}}{4}\right) + \mathcal{L}(4(\sqrt{5}-4))\right) = \frac{63}{10}. \tag{98}$$

$\mathbf{A}_1|\mathbf{E_8}$. The recurrence relations (74) read now

$$(Q^1)^2 = 1 + Q^3 \qquad (Q^2)^2 = 1 + Q^4 \qquad (Q^3)^2 = 1 + Q^1 Q^4 \qquad (Q^4)^2 = 1 + Q^3 Q^2 Q^5 \tag{99}$$

$$(Q^5)^2 = 1 + Q^4 Q^6 \qquad (Q^6)^2 = 1 + Q^5 Q^7$$
$$(Q^7)^2 = 1 + Q^6 Q^8 \qquad (Q^8)^2 = 1 + Q^7. \tag{100}$$

When setting $a = 1$ and $\tau = 1/32$, the E_8-characters (66)–(73) reduce to

$$\tilde{\chi}_{\lambda_1} = 1 \qquad \tilde{\chi}_{\lambda_8} = \sqrt{2} \qquad \tilde{\chi}_{\lambda_2} = \tilde{\chi}_{\lambda_3} = \tilde{\chi}_{\lambda_4} = \tilde{\chi}_{\lambda_5} = \tilde{\chi}_{\lambda_6} = \tilde{\chi}_{\lambda_7} = 0. \tag{101}$$

We may then identify them with combinations of A_1-characters

$$\tilde{\chi}_{\lambda_1} = \chi_{30\lambda} \qquad \tilde{\chi}_{\lambda_8} = \chi_{8\lambda} - \chi_{\lambda}. \tag{102}$$

With these numerical values we can express the solutions of (99) and (100) in terms of the E_8/A_1-characters

$$Q^1 = 2 + \tilde{\chi}_{\lambda_8} \qquad Q^2 = 3 + 2\tilde{\chi}_{\lambda_8} \qquad Q^3 = 5 + 4\tilde{\chi}_{\lambda_8} \qquad Q^4 = 4(4 + 3\tilde{\chi}_{\lambda_8}) \tag{103}$$

$$Q^5 = 3(3 + 2\tilde{\chi}_{\lambda_8}) \qquad Q^6 = 5 + 3\tilde{\chi}_{\lambda_8} \qquad Q^7 = 2 + 2\tilde{\chi}_{\lambda_8} \qquad Q^8 = \tilde{\chi}_{\lambda_1} + \tilde{\chi}_{\lambda_8}. \tag{104}$$

In [28] only the values for Q^1 and Q^8 were presented. As in the previous case, after relabelling our roots and translating to the x-variables we recover the numbers quoted in [14]. In this case

[3] There appears to be a small typo in equation (A.11.c) of [28], which reads when translated to our conventions, i.e. $6 \to 7$, $Q_7 = \chi_{\lambda_1}$ instead of (97).

the central charge (23) equals

$$c = \frac{6}{\pi^2}\left(\mathcal{L}\left(\sqrt{2}-\frac{1}{2}\right) + \mathcal{L}(12\sqrt{2}-16) + \mathcal{L}\left(\frac{40\sqrt{2}-8}{49}\right) + \mathcal{L}\left(\frac{12\sqrt{2}+15}{32}\right)\right.$$
$$\left. + \mathcal{L}\left(\frac{12\sqrt{2}-8}{9}\right) + \mathcal{L}\left(\frac{30\sqrt{2}+6}{49}\right) + \mathcal{L}\left(\frac{1}{4}+\frac{1}{\sqrt{2}}\right) + \mathcal{L}(2\sqrt{2}-2)\right) = \frac{15}{2}. \tag{105}$$

4.2.3. $\mathbf{D}_\ell|\mathbf{A}_{\tilde{\ell}}$. In this case the recurrence relations (74) read

$$\left(Q_a^k\right)^2 = Q_{a+1}^k Q_{a-1}^k + Q_a^{k+1} Q_a^{k-1} \qquad 1 \leqslant a \leqslant \ell - 3 \tag{106}$$

$$\left(Q_{\ell-2}^k\right)^2 = Q_{\ell-2}^{k-1} Q_{\ell-2}^{k+1} + Q_\ell^k Q_{\ell-1}^k Q_{\ell-3}^k \tag{107}$$

$$\left(Q_p^k\right)^2 = Q_{\ell-2}^k + Q_p^{k+1} Q_p^{k-1} \qquad p = \ell, \ell-1 \tag{108}$$

for $1 \leqslant k \leqslant \tilde{\ell}$. Also in this case we may exploit the symmetry of equations (74) in the two algebras. We simply have to exchange their roles in order to obtain a solution for the $\mathbf{D}_\ell|\mathbf{A}_{\tilde{\ell}}$-theory from the one for the $\mathbf{A}_{\tilde{\ell}}|\mathbf{D}_\ell$ reported in [27, 28]. Taking $\tau = 1/(2\ell + \tilde{\ell} - 1)$, we can express, following [27, 28], the Q's in terms of the PSW-characters of $\mathbf{D}_\ell$.

$$Q_s^k = \sum_{l_1=0}^{k} \cdots \sum_{l_{s-2}=0}^{k} \chi_{k\lambda_s + l_1(\lambda_1-\lambda_s)+\cdots+l_{s-2}(\lambda_{s-2}-\lambda_s)}(\tau) \tag{109}$$

$$Q_p^k = \sum_{\tilde{a}=0}^{k} \sum_{l_2=0}^{\tilde{a}} \cdots \sum_{l_{p-2}=0}^{\tilde{a}} \chi_{\tilde{a}\lambda_p + l_2(\lambda_2-\lambda_p)+\cdots+l_{p-2}(\lambda_{p-2}-\lambda_p)}(\tau) \tag{110}$$

$$Q_{\ell-1}^k = \chi_{k\lambda_{\ell-1}}(\tau) \qquad Q_\ell^k = \chi_{k\lambda_\ell}(\tau). \tag{111}$$

Here s and p are odd and even integers smaller than $\ell - 1$, respectively. Alternatively, we may also express the Q's in terms of the $A_{\tilde{\ell}}$-PSW-characters. For instance, for $\mathbf{D}_\ell|\mathbf{A}_2$ we find

$$Q_{2k}^1 = Q_{2k}^2 = 1 + \sum_{i=1}^{k}\left(\tilde{\chi}_{i\lambda} - \tilde{\chi}_{(i-2)\lambda} + \tilde{\chi}_{(\ell-i)\lambda} - \tilde{\chi}_{(\ell-i-2)\lambda}\right) \qquad 2k < \ell - 1 \tag{112}$$

$$Q_{2k-1}^1 = Q_{2k-1}^2 = \sum_{i=0}^{k-1}\left(\tilde{\chi}_{i\lambda} - \tilde{\chi}_{(i-2)\lambda}\right) + \sum_{i=1}^{k}\left(\tilde{\chi}_{(\ell-i)\lambda} - \tilde{\chi}_{(\ell-i-2)\lambda}\right) \qquad 2k < \ell \tag{113}$$

$$Q_\ell^1 = Q_{\ell-1}^1 = Q_\ell^2 = Q_{\ell-1}^2 = \tilde{\chi}_{\ell\lambda} - \tilde{\chi}_{(\ell-2)\lambda}. \tag{114}$$

We suppressed the τ-dependence, denote $\lambda = \lambda_1 = \lambda_2$ and recall that we take $\tilde{\chi}_{i\lambda} = 1$ for $i = 0$ and $\tilde{\chi}_{i\lambda} = 0$ for $i < 0$.

Let us now consider some theories which may not be obtained from the others previously studied, by exploiting the symmetry properties of the recurrence relations (74).

4.2.4. $\mathbf{D}_\ell|\mathbf{D}_{\tilde{\ell}}$. The recurrence relations (74) are now constructed from the symmetric D_l-incidence matrix, whose non-vanishing entries are

$$\hat{I}_{t,t+1} = 1 \qquad 1 \leqslant t \leqslant l-2 \qquad \hat{I}_{t,t-1} = 1 \qquad 2 \leqslant t \leqslant l-1 \qquad \hat{I}_{l,l-2} = 1 \tag{115}$$

such that $I = \hat{I}$ with $l = \ell$ and $\tilde{I} = \hat{I}$ with $l = \tilde{\ell}$.

$\boldsymbol{D}_4|\boldsymbol{D}_4$. For the choice $\tau = 1/12$ the D_4-characters (46) and (47) become

$$\chi_{\lambda_1} = 3+\sqrt{3} \qquad \chi_{2\lambda_1} = 5+3\sqrt{3} \qquad \chi_{3\lambda_1} = 6+4\sqrt{3} \qquad \chi_{\lambda_2} = 6+3\sqrt{3} \tag{116}$$

$$\chi_{2\lambda_2} = 15+9\sqrt{3} \qquad \chi_{3\lambda_2} = 10+6\sqrt{3} \qquad \chi_{\lambda_3} = \chi_{\lambda_1} \qquad \chi_{2\lambda_3} = \chi_{2\lambda_1} \qquad \chi_{3\lambda_3} = \chi_{3\lambda_1}. \tag{117}$$

The recurrence relations (74) are solved by

$$Q_1^1 = Q_1^3 = Q_1^4 = Q_3^1 = Q_4^1 = Q_3^3 = Q_4^3 = Q_3^4 = Q_4^4 = 4\chi_{\lambda_1} - \chi_{3\lambda_1} = 6 \tag{118}$$

$$Q_1^2 = Q_3^2 = Q_4^2 = Q_2^3 = Q_2^1 = Q_2^4 = 18 \qquad Q_2^2 = 108. \tag{119}$$

The central charge (23) in this case is simply

$$c = \frac{6}{\pi^2}\left(10\mathcal{L}\left(\frac{1}{2}\right) + 3\mathcal{L}\left(\frac{2}{3}\right) + 3\mathcal{L}\left(\frac{1}{3}\right)\right) = 8. \tag{120}$$

$\mathbf{D}_4|\mathbf{D_5}$. We now take $\tau = 1/14$ such that some of the D_4-characters (46) and (47) read

$$\chi_{\lambda_1} = \frac{\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}}{\sin\frac{\pi}{14}\sin\frac{3\pi}{14}} \qquad \chi_{2\lambda_1} = 2\chi_{\lambda_1}\cos\frac{\pi}{7}\sin\frac{5\pi}{14} = \frac{\chi_{4\lambda_1}}{2} \qquad \chi_{3\lambda_1} = \frac{\chi_{4\lambda_1}\cos^2\frac{3\pi}{7}}{\sin\frac{5\pi}{14}} \tag{121}$$

$$\chi_{\lambda_2} = \chi_{2\lambda_1} = \frac{\chi_{4\lambda_2}}{4} \qquad \chi_{2\lambda_2} = \frac{\chi_{2\lambda_1}\sin^2\frac{5\pi}{14}\sin\frac{2\pi}{7}}{\sin\frac{3\pi}{14}\sin^2\frac{\pi}{7}} \qquad \chi_{3\lambda_2} = \frac{\chi_{2\lambda_2}\sin^2\frac{3\pi}{7}}{\sin\frac{5\pi}{14}\sin^2\frac{2\pi}{7}} \tag{122}$$

and the ones for D_5

$$\tilde{\chi}_{\lambda_1} = \frac{\sin\frac{5\pi}{14}\sin\frac{3\pi}{7}}{\sin\frac{\pi}{14}\sin\frac{2\pi}{7}} \qquad \tilde{\chi}_{2\lambda_1} = \tilde{\chi}_{\lambda_1}\frac{\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}} \qquad \tilde{\chi}_{3\lambda_1} = \chi_{2\lambda_1} \tag{123}$$

$$\tilde{\chi}_{\lambda_2} = \tilde{\chi}_{\lambda_1}\frac{\sin\frac{5\pi}{14}\sin\frac{2\pi}{7}}{\sin\frac{3\pi}{14}\sin\frac{\pi}{7}} \qquad \tilde{\chi}_{2\lambda_2} = 2\tilde{\chi}_{2\lambda_1}\frac{\cos\frac{3\pi}{14}}{\sin\frac{\pi}{7}} \qquad \tilde{\chi}_{3\lambda_2} = \frac{(\tilde{\chi}_{\lambda_2})^2}{\tilde{\chi}_{\lambda_1}} \tag{124}$$

$$\tilde{\chi}_{\lambda_3} = 2\tilde{\chi}_{\lambda_2}\cos\frac{\pi}{7} \qquad \tilde{\chi}_{2\lambda_3} = \frac{\tilde{\chi}_{3\lambda_2}}{2} \qquad \tilde{\chi}_{3\lambda_3} = \frac{\tilde{\chi}_{2\lambda_1}}{2} \tag{125}$$

$$\tilde{\chi}_{\lambda_4} = \frac{\sin^2\frac{3\pi}{7}}{\sin\frac{\pi}{14}\sin\frac{3\pi}{14}} \qquad \tilde{\chi}_{2\lambda_4} = \tilde{\chi}_{2\lambda_1}\frac{1}{\sin\left(\frac{3\pi}{14}\right)} \qquad \tilde{\chi}_{3\lambda_4} = \chi_{3\lambda_1}. \tag{126}$$

We may then express the PSW-characters of D_5 in terms of characters of D_4,

$$\begin{aligned}
&\tilde{\chi}_{\lambda_1} = \left(\chi_{3\lambda_1} - \chi_{\lambda_2}\right)/2 \qquad \tilde{\chi}_{2\lambda_1} = \left(\chi_{3\lambda_1} - 2\right)/2 \qquad \tilde{\chi}_{3\lambda_1} = \chi_{2\lambda_1}\\
&\tilde{\chi}_{\lambda_2} = \left(\chi_{3\lambda_1} + \chi_{\lambda_2} - 2\chi_{\lambda_1} - 2\right)/2 \qquad \tilde{\chi}_{2\lambda_2} = \left(\chi_{3\lambda_2} - \chi_{3\lambda_1}\right)/2\\
&\tilde{\chi}_{3\lambda_2} = \left(-10\chi_{\lambda_1} + 9\left(\chi_{2\lambda_1} - 1\right) + 6\chi_{3\lambda_1} - \chi_{2\lambda_2}\right)/2\\
&\tilde{\chi}_{\lambda_3} = \chi_{3\lambda_1} - 1 \qquad \tilde{\chi}_{2\lambda_3} = 2\chi_{4\lambda_1} \qquad \tilde{\chi}_{3\lambda_3} = \left(\chi_{3\lambda_1} - 2\right)/2\\
&\tilde{\chi}_{\lambda_4} = \left(\chi_{3\lambda_1} - 2\chi_{\lambda_1}\right)/2 \qquad \tilde{\chi}_{2\lambda_4} = \chi_{3\lambda_1} - \chi_{\lambda_1} - 1 \qquad \tilde{\chi}_{3\lambda_4} = \chi_{3\lambda_1}.
\end{aligned} \tag{127}$$

In terms of these quantities we may then solve the recurrence relations by

$$\begin{aligned}
&Q_1^1 = 1 + \chi_{\lambda_1}\\
&Q_1^2 = 6\left(\chi_{3\lambda_1} + \chi_{3\lambda_2} + 1\right) - 10\left(\chi_{\lambda_1} + \chi_{2\lambda_1} + \chi_{4\lambda_1}\right) + 4\chi_{2\lambda_2} - 9\chi_{4\lambda_2}\\
&Q_1^3 = 2\left(2 - \chi_{\lambda_1} + \chi_{2\lambda_1} + \chi_{2\lambda_2} - \chi_{3\lambda_2} + \chi_{4\lambda_2}\right)\\
&Q_1^4 = 10\left(\chi_{2\lambda_2} - \chi_{3\lambda_1} - \chi_{4\lambda_1} - \chi_{4\lambda_2}\right) - 8\chi_{\lambda_1} - 5\chi_{2\lambda_1} + 6\chi_{3\lambda_2} - 7\\
&Q_2^1 = 8\left(\chi_{2\lambda_2} - \chi_{\lambda_1} - \chi_{4\lambda_1} - \chi_{4\lambda_2} + 1\right) + 5\left(\chi_{3\lambda_1} - \chi_{2\lambda_1}\right) + 2\chi_{3\lambda_2}\\
&Q_2^2 = 8\left(\chi_{3\lambda_2} + \chi_{3\lambda_1} - \chi_{\lambda_1} - \chi_{4\lambda_2}\right) - 5\chi_{4\lambda_1} - 4\chi_{2\lambda_2} + 2
\end{aligned} \tag{128}$$

$$
\begin{aligned}
&Q_2^3 = 6\left(\chi_{\lambda_1} + \chi_{4\lambda_1} + \chi_{4\lambda_2}\right) + 4\chi_{2\lambda_1} - 2\chi_{3\lambda_1} + 1\\
&Q_2^4 = Q_2^5 = 6\left(\chi_{4\lambda_2} - \chi_{2\lambda_2}\right) + 4\left(\chi_{\lambda_1} + \chi_{4\lambda_1}\right) - \chi_{3\lambda_2} + \chi_{3\lambda_1}\\
&Q_3^1 = Q_4^1 = Q_1^1 \qquad Q_3^2 = Q_4^2 = Q_1^2 \qquad Q_3^4 = Q_3^5 = Q_1^4 = Q_4^4 = Q_4^5\\
&Q_3^3 = Q_4^3 = Q_1^3.
\end{aligned}
$$

Using these values we compute numerically the central charge as $c = 80/7$.

$\boldsymbol{D}_5|\boldsymbol{D}_5$. For $\tau = 1/16$ and $\ell = 5$, the D_5-PSW-characters (46) and (67) become

$$
\begin{aligned}
&\chi_{\lambda_1} = \sqrt{2}\frac{\sin\frac{5\pi}{16}}{\sin\frac{\pi}{16}} \qquad \chi_{2\lambda_1} = 4 + 3\sqrt{2} + 2\sqrt{10 + 7\sqrt{2}}\\
&\chi_{3\lambda_1} = 8 + 5\sqrt{2} + \sqrt{2(58 + 41\sqrt{2})} \qquad \chi_{4\lambda_1} = 2\chi_{2\lambda_1}\\
&\chi_{\lambda_2} = \chi_{2\lambda_1} + 1 \qquad \chi_{2\lambda_2} = 22 + 17\sqrt{2} + 2\sqrt{274 + 193\sqrt{2}}\\
&\chi_{3\lambda_2} = 46 + 32\sqrt{2} + 6\sqrt{116 + 82\sqrt{2}} \qquad \chi_{4\lambda_2} = 4 + 6\chi_{2\lambda_1}\\
&\chi_{\lambda_3} = 2 + 2\chi_{\lambda_2} \qquad \chi_{2\lambda_3} = 61 + 41\sqrt{2} + 6\sqrt{194 + 137\sqrt{2}}\\
&\chi_{3\lambda_3} = 100 + 69\sqrt{2} + 13\sqrt{116 + 82\sqrt{2}} \qquad \chi_{4\lambda_3} = \chi_{4\lambda_2}\\
&\chi_{\lambda_4} = 2(1 + \sqrt{2} + \sqrt{2 + \sqrt{2}}) \qquad \chi_{2\lambda_4} = \chi_{3\lambda_1} \qquad \chi_{3\lambda_4} = 2\chi_{3\lambda_1}\\
&\chi_{4\lambda_4} = 18 + 14\sqrt{2} + 6\sqrt{20 + 14\sqrt{2}}.
\end{aligned}
\tag{129}
$$

Noting the symmetry $Q_a^i = Q_i^a$, we may now express the Q's in terms of D_5-characters:

$$
\begin{aligned}
&Q_1^1 = 2\left(\chi_{\lambda_2} - \chi_{\lambda_1} - \chi_{\lambda_4}\right)\\
&Q_1^2 = 2\left(\chi_{2\lambda_3} + \chi_{\lambda_4} - \chi_{\lambda_1} - \chi_{2\lambda_1} - \chi_{3\lambda_1} - \chi_{2\lambda_2}\right) - \chi_{4\lambda_1} - \chi_{3\lambda_4} - \chi_{4\lambda_4}\\
&Q_1^3 = 2\left(\chi_{3\lambda_3} - \chi_{\lambda_1} - \chi_{2\lambda_1} - \chi_{3\lambda_1} - \chi_{4\lambda_1} - \chi_{2\lambda_2} - \chi_{\lambda_4} - \chi_{4\lambda_4}\right) + \chi_{3\lambda_4} - \chi_{\lambda_2} - \chi_{2\lambda_3}\\
&Q_1^4 = Q_1^5 = \chi_{4\lambda_1} + \chi_{3\lambda_2} + \chi_{2\lambda_3} - \chi_{\lambda_1} - \chi_{2\lambda_1} - \chi_{\lambda_2} - \chi_{3\lambda_3} - \chi_{\lambda_4}\\
&Q_2^2 = 2\left(\chi_{4\lambda_1} + \chi_{2\lambda_2} + \chi_{2\lambda_3} + \chi_{\lambda_4} + \chi_{4\lambda_4} - \chi_{\lambda_1} - \chi_{2\lambda_1} - \chi_{3\lambda_1} - \chi_{4\lambda_2} - \chi_{3\lambda_4}\right)\\
&\qquad + \chi_{3\lambda_2} - \chi_{\lambda_2} - \chi_{3\lambda_3}\\
&Q_2^3 = 2\left(\chi_{3\lambda_2} + \chi_{3\lambda_3} + \chi_{3\lambda_4} - \chi_{\lambda_1} - \chi_{2\lambda_1} - \chi_{4\lambda_1} - \chi_{2\lambda_2} - \chi_{\lambda_4} - \chi_{4\lambda_4}\right)\\
&\qquad + \chi_{2\lambda_3} - \chi_{3\lambda_1} - \chi_{\lambda_2} - 1\\
&Q_2^4 = Q_2^5 = 1 + \chi_{\lambda_1} + \chi_{4\lambda_1} + \chi_{3\lambda_2} + \chi_{2\lambda_3} + \chi_{4\lambda_4} - \chi_{2\lambda_2} - \chi_{3\lambda_3}\\
&Q_3^3 = 8\left(\chi_{2\lambda_3} + \chi_{3\lambda_3} + \chi_{4\lambda_4} - \chi_{\lambda_1} - \chi_{2\lambda_1} - \chi_{3\lambda_1} - \chi_{4\lambda_1} - \chi_{\lambda_2} - \chi_{2\lambda_2}\right)\\
&\qquad + 7\left(\chi_{3\lambda_2} + \chi_{4\lambda_2} + \chi_{3\lambda_4}\right) - 5\chi_{\lambda_4} + 4\\
&Q_3^4 = Q_3^5 = \chi_{4\lambda_1} + \chi_{3\lambda_2} + \chi_{2\lambda_3} - \chi_{2\lambda_1} - \chi_{\lambda_2} - \chi_{2\lambda_2} - \chi_{4\lambda_2} - \chi_{4\lambda_4}\\
&Q_5^5 = Q_4^4 = Q_4^5 = 1 + \chi_{\lambda_1} + \chi_{4\lambda_1} + \chi_{\lambda_2} + \chi_{2\lambda_2} + \chi_{4\lambda_4} - \chi_{3\lambda_1} - \chi_{3\lambda_2}.
\end{aligned}
\tag{130}
$$

Using these values we compute numerically the central charge as $c = 25/2$.

4.2.5. $D_4|E_6$. In this case recurrence relations (74) read

$$
\left(Q_1^1\right)^2 = Q_2^1 + Q_1^2 \qquad \left(Q_1^2\right)^2 = Q_2^2 + Q_1^4 \qquad \left(Q_1^3\right)^2 = Q_2^3 + Q_1^4 Q_1^1 \tag{131}
$$

$$
\left(Q_1^4\right)^2 = Q_2^4 + Q_1^2\left(Q_1^3\right)^2 \qquad \left(Q_2^1\right)^2 = Q_4^1 + Q_2^2 \qquad \left(Q_2^2\right)^2 = Q_4^2 + Q_2^4 \tag{132}
$$

$$
\left(Q_2^3\right)^2 = Q_4^3 + Q_2^4 Q_2^1 \qquad \left(Q_2^4\right)^2 = Q_4^4 + Q_2^2\left(Q_2^3\right)^2. \tag{133}
$$

We already took the relations

$$Q_a^1 = Q_a^6 \qquad Q_a^3 = Q_a^5 \qquad Q_1^i = Q_3^i = Q_4^i \qquad 1 \leqslant a \leqslant 4, 1 \leqslant i \leqslant 6 \tag{134}$$

into account which arise as a consequence of the symmetries of the D_4 and E_6 Dynkin diagrams. Taking now $\tau = 1/18$, $\ell = 4$ and $\tilde{\ell} = 6$, the D_4-characters turn out to be

$$\begin{aligned}
&\chi_{\lambda_1} = \sqrt{3}\frac{\sin\frac{2\pi}{9}}{\sin\frac{\pi}{18}} \qquad \chi_{2\lambda_1} = \sqrt{3}\frac{\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}}{\sin\frac{\pi}{18}\sin\frac{\pi}{9}} \qquad \chi_{3\lambda_1} = \sqrt{3}\chi_{2\lambda_1}\frac{\sin\frac{4\pi}{9}}{\sin\frac{5\pi}{18}}\\
&\chi_{4\lambda_1} = \frac{2}{\sqrt{3}}\chi_{3\lambda_1}\frac{\sin\frac{7\pi}{18}}{\sin\frac{2\pi}{9}} \qquad \chi_{5\lambda_1} = \chi_{4\lambda_1}\frac{\sin^2\frac{4\pi}{9}}{\sin\frac{7\pi}{18}\sin\frac{5\pi}{18}} \qquad \chi_{6\lambda_1} = \frac{2}{\sqrt{3}}\chi_{5\lambda_1}\frac{\sin\frac{7\pi}{18}}{\sin\frac{4\pi}{9}}\\
&\chi_{\lambda_2} = \frac{1}{6}\chi_{3\lambda_1}\frac{\tan\frac{2\pi}{9}}{\sin\frac{\pi}{9}} \qquad \chi_{2\lambda_2} = \frac{2}{3}\chi_{3\lambda_2}\frac{\sin^2\frac{2\pi}{9}}{\sin^2\frac{7\pi}{18}} \qquad \chi_{3\lambda_2} = \sqrt{3}\chi_{2\lambda_1}^2\frac{\sin\frac{\pi}{18}}{\sin\frac{\pi}{9}}\\
&\chi_{4\lambda_2} = \frac{\sqrt{3}}{2}\chi_{4\lambda_1}\chi_{2\lambda_1}\frac{\sin\frac{\pi}{18}}{\sin\frac{\pi}{9}} \qquad \chi_{5\lambda_2} = \frac{2}{3}\chi_{4\lambda_2}\frac{\sin^2\frac{4\pi}{9}}{\sin^2\frac{5\pi}{18}} \qquad \chi_{6\lambda_2} = \frac{4}{\sqrt{3}}\chi_{5\lambda_2}\frac{\sin\frac{4\pi}{9}\sin\frac{\pi}{18}}{\sin^2\frac{7\pi}{18}}
\end{aligned} \tag{135}$$

and the E_6-characters are

$$\tilde{\chi}_{\lambda_1} = \frac{\sqrt{3}}{2\sin\frac{2\pi}{9}\sin\frac{\pi}{18}} \qquad \tilde{\chi}_{\lambda_2} = \frac{4}{\sqrt{3}}\tilde{\chi}_{\lambda_1}\sin\frac{5\pi}{18}\sin\frac{4\pi}{9} \qquad \tilde{\chi}_{\lambda_3} = \frac{3}{4}\tilde{\chi}_{\lambda_2}\frac{\cos\frac{\pi}{9}}{\cos\frac{2\pi}{9}} \tag{136}$$

$$\tilde{\chi}_{\lambda_4} = \frac{8}{3}\tilde{\chi}_{\lambda_2}\tilde{\chi}_{\lambda_3}\frac{\sin\frac{\pi}{18}\cos\frac{\pi}{9}}{\sin\frac{7\pi}{18}} \qquad \tilde{\chi}_{2\lambda_1} = 4\tilde{\chi}_{\lambda_1}\cos\frac{2\pi}{9}\cos\frac{\pi}{9} \qquad \tilde{\chi}_{2\lambda_2} = 2\tilde{\chi}_{2\lambda_1} \tag{137}$$

$$\tilde{\chi}_{2\lambda_3} = 2\sqrt{3}\tilde{\chi}_{3\lambda_1}\frac{\cos\frac{2\pi}{9}}{\sin\frac{5\pi}{18}} \qquad \tilde{\chi}_{2\lambda_4} = 36\tilde{\chi}_{3\lambda_2}^2\cos^2\frac{2\pi}{9} \qquad \tilde{\chi}_{3\lambda_1} = 2\frac{\sin\frac{4\pi}{9}\sin\frac{7\pi}{18}}{\sin\frac{\pi}{18}\sin\frac{\pi}{9}} \tag{138}$$

$$\tilde{\chi}_{3\lambda_2} = \tilde{\chi}_{3\lambda_3} = \tilde{\chi}_{3\lambda_1} + 2 = \frac{\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}}{\sin\frac{4\pi}{9}\sin\frac{\pi}{9}} \qquad \tilde{\chi}_{3\lambda_4} = \tilde{\chi}_{2\lambda_4} \tag{139}$$

such that we find the following relations amongst them:

$$\begin{aligned}
&\tilde{\chi}_{\lambda_1} = 1 - 2(\chi_{\lambda_1} - \chi_{5\lambda_1} - \chi_{\lambda_2} + \chi_{2\lambda_2} - \chi_{3\lambda_2} + \chi_{6\lambda_2}) - \chi_{3\lambda_1}\\
&\tilde{\chi}_{\lambda_2} = 2(\chi_{5\lambda_1} - \chi_{2\lambda_1} - \chi_{2\lambda_2} - \chi_{6\lambda_2}) + \chi_{5\lambda_2}\\
&\tilde{\chi}_{\lambda_3} = 2(\chi_{\lambda_1} - \chi_{2\lambda_1} + \chi_{5\lambda_1} - \chi_{2\lambda_2} - \chi_{3\lambda_2} + \chi_{5\lambda_2} - \chi_{6\lambda_2} + 1)\\
&\tilde{\chi}_{\lambda_4} = 2\left(1 - \chi_{\lambda_1} - \chi_{2\lambda_1} - \chi_{3\lambda_1} + \chi_{4\lambda_1} + \chi_{5\lambda_1} + \chi_{\lambda_2} - \chi_{2\lambda_2} + \chi_{3\lambda_2} - \chi_{6\lambda_2}\right)\\
&\tilde{\chi}_{2\lambda_1} = 2\left(\chi_{5\lambda_1} - \chi_{\lambda_1} - \chi_{2\lambda_1} - \chi_{3\lambda_1}\right) + \chi_{4\lambda_1} - \chi_{2\lambda_2}\\
&\tilde{\chi}_{2\lambda_3} = 2\left(\chi_{5\lambda_1} - \chi_{\lambda_1} - \chi_{2\lambda_1} - \chi_{3\lambda_1}\right) - \chi_{4\lambda_1} + \chi_{2\lambda_2}\\
&\tilde{\chi}_{2\lambda_4} = 2\left(1 - \chi_{\lambda_1} - \chi_{2\lambda_1}\right) - \chi_{3\lambda_1} + \chi_{4\lambda_1} + \chi_{6\lambda_1}\\
&\tilde{\chi}_{3\lambda_1} = 2\left(\chi_{4\lambda_1} + \chi_{6\lambda_1} - \chi_{\lambda_1} - \chi_{2\lambda_1} - \chi_{3\lambda_1} - \chi_{5\lambda_1}\right)\\
&\tilde{\chi}_{3\lambda_2} = \tilde{\chi}_{3\lambda_3} = 2 + \tilde{\chi}_{3\lambda_1}.
\end{aligned} \tag{140}$$

The recurrence relations (131)–(133) are then solved by

$$\begin{aligned}
&Q_1^1 = 2\chi_{\lambda_2} - \chi_{\lambda_1} - \chi_{2\lambda_1}\\
&Q_1^2 = \chi_{5\lambda_1} + \chi_{\lambda_2} + \chi_{6\lambda_2} - \chi_{\lambda_1} - \chi_{2\lambda_1} - \chi_{3\lambda_1} - \chi_{3\lambda_2}\\
&Q_1^3 = \chi_{\lambda_1} + \chi_{2\lambda_1} + \chi_{6\lambda_1} + \chi_{4\lambda_2} - \chi_{5\lambda_2} - 1\\
&Q_1^4 = 1 - \chi_{4\lambda_2} - 2\left(\chi_{\lambda_1} + \chi_{2\lambda_1} + \chi_{3\lambda_1} + \chi_{4\lambda_1} + \chi_{5\lambda_1} - \chi_{6\lambda_1} + \chi_{2\lambda_2} - \chi_{5\lambda_2}\right)\\
&Q_2^1 = 1 + \chi_{\lambda_1} + \chi_{2\lambda_1} + \chi_{4\lambda_1} - \chi_{5\lambda_1} + \chi_{2\lambda_2} + \chi_{4\lambda_2} - \chi_{5\lambda_2}\\
&Q_2^2 = \chi_{\lambda_1} + \chi_{2\lambda_1} - \chi_{3\lambda_1} - \chi_{4\lambda_1} + \chi_{5\lambda_1} + \chi_{6\lambda_1} + \chi_{\lambda_2} + \chi_{3\lambda_2} - \chi_{4\lambda_2} + \chi_{6\lambda_2} - 1
\end{aligned} \tag{141}$$

$$Q_2^3 = 2\big(\chi_{3\lambda_2} + \chi_{5\lambda_2} - \chi_{\lambda_1} - \chi_{2\lambda_1} - \chi_{3\lambda_1} - \chi_{4\lambda_1} - \chi_{6\lambda_2}\big) - \chi_{5\lambda_1} - \chi_{6\lambda_1} - \chi_{2\lambda_2}$$
$$Q_2^4 = 8\big(\chi_{3\lambda_2} + \chi_{4\lambda_2} + \chi_{5\lambda_2} - \chi_{\lambda_1} - \chi_{2\lambda_1} - \chi_{3\lambda_1} - \chi_{4\lambda_1} - \chi_{5\lambda_1} - \chi_{\lambda_2} - 1\big) - 7\chi_{6\lambda_1} + 6\chi_{6\lambda_1}.$$

Using these values we compute numerically the central charge as $c = 16$.

5. Unstable quasi-particles

Once a character is expressed in the generic form (4), it does not only allow a derivation of the constant TBA equations, but also, when interpreted as a partition function, one may construct quasi-particle spectra of different statistical nature. We proceed in the usual fashion, but we will now introduce as the main novelty also unstable quasi-particles inside the spectrum. As usual [7] we parametrize the partition function $\chi(q = e^{2\pi v/ktL})$ by Boltzmann's constant k, the temperature T, the size of the quantizing system L and the speed of sound v. We then equate it with $\sum_{n=0}^{\infty} P(E_n)\exp(-E_n/kT)$, where $P(E_n)$ denotes the degeneracy of the particular energy level $E_n = E_n(p_A)$ as a function of the single particle contributions of type A. It is the aim in this analysis to identify the spectrum expressed in terms of the p_A. Technically this can be achieved by making use of the expressions for the number of partitions $\mathcal{Q}_s(n,m)$ ($\mathcal{P}_s(n,m)$) of the positive integer n into m non-negative (distinct) integers smaller or equal to s (see e.g. [8]):

$$\sum_{n=0}^{\infty} \mathcal{P}_s(n,m)q^n = q^{m(m-1)} \begin{bmatrix} s+1 \\ m \end{bmatrix}_q \qquad \sum_{n=0}^{\infty} \mathcal{Q}_s(n,m)q^n = \begin{bmatrix} s+m \\ m \end{bmatrix}_q . \tag{142}$$

Introducing in the standard way [7] some internal quantum numbers, we construct for instance (in units of $2\pi/L$) a purely fermionic

$$p^a_{N_a}(\vec{k}) = \tfrac{1}{2}([M_{ab}]_q - \delta_{ab})k_b + \tfrac{1}{2} + B_a + N_a \tag{143}$$

or purely bosonic

$$p^a_{N_a}(\vec{k}) = \tfrac{1}{2}[M_{ab}]_q k_b + B_a + \hat{N}_a \tag{144}$$

quasi-particle spectrum. The positive integers N_a and $\hat{N}_a$ are constrained from above as $N_a < \mathrm{Int}((1 - [M_{ab}]_q)\ k_b + B'_a)$ and $\hat{N}_a \leqslant \mathrm{Int}((1 - [M_{ab}]_q)\ k_b + m_a + B'_a)$, with $\mathrm{Int}(x)$ to be the integer part of x. Like in the non-deformed case, it is of course also possible to construct spectra related to more exotic or even with mixed statistics.

We now expect that at a certain energy scale some unstable particles vanish from the spectrum. The mechanism for this is that the upper bounds $N_a, \hat{N}_a$ involved in the expressions for the possible momenta $p^a_{N_a}(\vec{k})$, $p^a_{\hat{N}_a}(\vec{k})$ decrease. We illustrate this with some examples. Denoting the character for the vacuum sector of the minimal model $\mathcal{M}(k, k+1)$ by $\chi^k(q)$ [35], we compute for instance

$$\chi^2(q) - \chi^1(q) = q^6 + q^7 + 2q^8 + 3q^9 + 5q^{10} + 6q^{11} + 9q^{12} + 11q^{13} + 16q^{14} + 20q^{15} + 27q^{16} + 33q^{17} + 44q^{18} + 54q^{19} + 70q^{20} + \mathcal{O}(q^{21}). \tag{145}$$

This means, for example, comparing $\chi^1(q)$ and $\chi^2(q)$ one particle should vanish from the spectrum of $\mathcal{M}(2,3)$ at level 6 when we vary the value of the resonance parameter such that it flows to $\mathcal{M}(1,2)$. Indeed, in the purely fermionic spectrum, we have the possibility of a six-particle contribution involving four particles of type 1 and two of type 2 with $N_2 < \mathrm{Int}\big(2\big[(1 - \exp(-r/2m_2)) + \exp\big(-r/2e^{|\sigma_{12}|/2}\big)\big]\big)$. This means for $rm_2/2 \ll 1$ and $r/2e^{|\sigma_{12}|/2} \ll 1$ the state

$$\big|p^1_0(4,2),\, p^1_1(4,2),\, p^1_2(4,2),\, p^1_3(4,2),\, p^2_0(4,2),\, p^2_1(4,2)\big\rangle \tag{146}$$

is allowed. It is then clear that when we increase σ_{12}, this state disappears from the spectrum. At the same time the state

$$\left|p_0^1(4,2),\, p_1^1(4,2),\, p_2^1(4,2),\, p_4^1(4,2),\, p_0^2(4,2),\, p_1^2(4,2)\right\rangle \tag{147}$$

at level 7 and the two states

$$\left|p_0^1(4,2),\, p_1^1(4,2),\, p_2^1(4,2),\, p_5^1(4,2),\, p_0^2(4,2),\, p_1^2(4,2)\right\rangle \tag{148}$$

$$\left|p_0^1(4,2),\, p_1^1(4,2),\, p_3^1(4,2),\, p_4^1(4,2),\, p_0^2(4,2),\, p_1^2(4,2)\right\rangle \tag{149}$$

at level 8, etc vanish for the same reason.

6. Conclusions

We have demonstrated that it is possible to construct scaling functions which reproduce the renormalization group flow by q-deforming fermionic versions of Virasoro characters in a very natural way. We investigated a fairly generic class of theories related to a pair of simple simply laced Lie algebras $\mathbf{g}$ and $\tilde{\mathbf{g}}$ or associated coset models. The construction procedure relies on the fact that the characters, quantities of the massless theory, involve data of the massive theory, i.e. the phases of the S-matrices. At the fixed points of these flows we solved the relevant recurrence relations analytically in terms of PSW-characters. We provided here various new solutions for particular choices of the algebras involved. It would be extremely interesting to answer the question whether it is possible to solve these relations in a completely generic, i.e. case-independent fashion. One should note that our solutions admit various ambiguities, i.e. the sums are not unique since there are numerous character identities involved or they might be expressed in terms of direct products of characters in a Clebsch–Gordan sense. This arbitrariness might be eliminated when one possibly finds a deeper interpretation of the recurrence relation in terms of representation theory.

Furthermore, it would be interesting to investigate whether it is possible to modify the PSW-characters, for instance by a specific choice of the τ's, in such a way that they solve the full r-dependent recurrence relations (22) exactly. Noting that our scaling functions only coincide qualitatively with those obtained from the full TBA analysis, in the sense that they have the plateaux precisely in the same position, including their size in the r-direction, one may ask a stronger question: is it possible to find versions of PSW-characters such that the full TBA equations, i.e. their formulation in terms of so-called Y-systems (see e.g. [34]), are reproduced?

The functions we constructed allow for a far easier investigation of the RG-behaviour than the full TBA-system [2], the scaled c-theorem [3, 4] or the semiclassical analysis [5]. This allows us to investigate systems of more complex nature such as $A_1|E_6$ or flows between different supersymmetric series. It would be interesting to investigate the latter flow in the other approaches.

The level-rank duality of type (12) gives a hint why it is possible to obtain the same flow by means of a theory involving unstable particles and alternatively as massless flows in the sense of [19]. The concrete link, however, i.e. the question of how this duality is reflected in the massive models, that is the scattering matrix, still eludes our analysis.

We have also shown that our q-deformed characters allow for the construction of spectra, which also involve unstable quasi-particles. The 'decay' of these particles from the spectrum is governed by a variable bound on the momenta depending on the resonance parameter.

Concerning the specific theories investigated, it would be of interest to extend the analysis to models which also involve non-simply laced algebras, albeit for $\mathbf{g}$ non-simply laced consistent S-matrices have not been constructed at present.

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